Applications of the Wronskian to ordinary linear differential equations

Consider a set of \( n \) continuous functions \( y_i(x) \), \( i = 1, 2, 3, \ldots, n \), each of which is differentiable at least \( n \) times. Then if there exist a set of constants \( \lambda_i \) that are not all zero such that

\[
\lambda_1 y_1(x) + \lambda_2 y_2(x) + \cdots + \lambda_n y_n(x) = 0, \tag{1}
\]

then we say that the set of functions \( \{y_i(x)\} \) are linearly dependent. If the only solution to eq. (1) is \( \lambda_i = 0 \) for all \( i \), then the set of functions \( \{y_i(x)\} \) are linearly independent.

The Wronskian matrix is defined as:

\[
\Phi[y_i(x)] = \begin{pmatrix}
y_1 & y_2 & \cdots & y_n \\
y'_1 & y'_2 & \cdots & y'_n \\
y''_1 & y''_2 & \cdots & y''_n \\
\vdots & \vdots & \ddots & \vdots \\
y^{(n-1)}_1 & y^{(n-1)}_2 & \cdots & y^{(n-1)}_n
\end{pmatrix},
\]

where

\[
y'_i \equiv \frac{dy_i}{dx}, \quad y''_i \equiv \frac{d^2y_i}{dx^2}, \quad \ldots, \quad y^{(n-1)}_i \equiv \frac{d^{(n-1)}y_i}{dx^{n-1}}.
\]

The Wronskian is defined to be the determinant of the Wronskian matrix,

\[
W(x) \equiv \det \Phi[y_i(x)]. \tag{2}
\]

In Physics 116A, we learned that if \( \{y_i(x)\} \) is a linearly dependent set of functions then the Wronskian must vanish. However, the converse is not necessarily true, as one can find cases in which the Wronskian vanishes without the functions being linearly independent.

Nevertheless, if the \( y_i(x) \) are solutions to an \( n \)th order ordinary linear differential equation, then the converse does hold. That is, if the \( y_i(x) \) are solutions to an \( n \)th order ordinary linear differential equation and the Wronskian of the \( y_i(x) \) vanishes, then \( \{y_i(x)\} \) is a linearly dependent set of functions. Moreover, if the Wronskian does not vanish for some value of \( x \), then it is does not vanish for all values of \( x \), in which case an arbitrary linear combination of the \( y_i(x) \) constitutes the most general solution to the \( n \)th order ordinary linear differential equation.

To demonstrate that the Wronskian either vanishes for all values of \( x \) or it is never equal to zero, if the \( y_i(x) \) are solutions to an \( n \)th order ordinary linear differential equation, we shall derive a formula for the Wronskian. Consider the differential equation,

\[
a_0(x)y^{(n)} + a_1(x)y^{(n-1)} + \cdots + a_{n-1}(x)y' + a_n(x)y = 0. \tag{3}
\]
We are interested in solving this equation over an interval of the real axis \( a < x < b \) in which \( a_0(x) \neq 0 \). We can rewrite eq. (3) as a first order matrix differential equation. Defining the vector
\[
\vec{Y} = \begin{pmatrix} y \\ y' \\ y'' \\ \vdots \\ y^{(n-1)} \end{pmatrix},
\]
It is straightforward to verify that eq. (3) is equivalent to
\[
\frac{d\vec{Y}}{dx} = A(x)\vec{Y},
\]
where the matrix \( A(x) \) is given by
\[
A(x) = \begin{pmatrix}
0 & 1 & 0 & 0 & \cdots & 0 \\
0 & 0 & 1 & 0 & \cdots & 0 \\
0 & 0 & 0 & 1 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & 0 & \cdots & 1 \\
\frac{-a_n(x)}{a_0(x)} & \frac{-a_{n-1}(x)}{a_0(x)} & \frac{-a_{n-2}(x)}{a_0(x)} & \frac{-a_{n-3}(x)}{a_0(x)} & \cdots & \frac{-a_1(x)}{a_0(x)} \\
\end{pmatrix}.
\]
It immediately follows that if the \( y_i(x) \) are linearly independent solutions to eq. (3), then the Wronskian matrix satisfies the first order matrix differential equation,
\[
\frac{d\Phi}{dx} = A(x)\Phi.
\]
Using eq. (21) of Appendix I, it follows that
\[
\frac{d}{dx} \det \Phi = \det \Phi \text{Tr} \left( \Phi^{-1} \frac{d\Phi}{dx} \right) = \det \Phi \text{Tr} A(x),
\]
after using eq. (5) and the cyclicity property of the trace (i.e. the trace is unchanged by cyclically permuting the matrices inside the trace). In terms of the Wronskian \( W \) defined in eq. (2),
\[
\frac{dW}{dx} = W \text{Tr} A(x).
\]
This is a first order differential equation for \( W \) that is easily integrated,
\[
W(x) = W(x_0) \exp \left\{ \int_{x_0}^x \text{Tr} A(t)dt \right\}.
\]
Using eq. (4), it follows that \( \text{Tr} A(t) = -a_1(t)/a_0(t) \). Hence, we arrive at Liouville’s formula (also called Abel’s formula),
\[
W(x) = W(x_0) \exp \left\{ -\int_{x_0}^x \frac{a_1(t)}{a_0(t)}dt \right\}.
\]
Note that if \( W(x_0) \neq 0 \), then the result for \( W(x) \) is strictly positive or strictly negative depending on the sign of \( W(x_0) \). This confirms our assertion that the Wronskian either vanishes for all values of \( x \) or it is never equal to zero.

Let us apply these results to an ordinary second order linear differential equation,

\[
y'' + a(x)y' + b(x)y = 0,
\]

where for convenience, we have divided out by the function that originally appeared multiplied by \( y'' \). Then, eq. (7) yields the Wronskian, which we shall write in the form:

\[
W(x) = c \exp \left\{ - \int a(x)dx \right\},
\]

where \( c \) is an arbitrary nonzero constant and \( \int a(x)dx \) is the indefinite integral of \( a(x) \). A simpler and more direct derivation of eq. (9) is presented in Appendix II.

The Wronskian also appears in the following application. Suppose that one of the two solutions of eq. (8), denoted by \( y_1(x) \) is known. Then, one can derive a second linearly independent solution \( y_2(x) \) by the method of variations of parameters.* In this context, the idea of this method is to define a new variable \( v \),

\[
y_2(x) = v(x)y_1(x) = y_1(x) \int w(x)dx,
\]

where

\[
v' \equiv w.
\]

Then, we have

\[
y_2' = vy_1' + wy_1, \quad y_2'' = vy_1'' + w'y_1 + 2wy_1'.
\]

Since \( y_2 \) is a solution to eq. (8), it follows that

\[
w'y_1 + w[2y_1' + a(x)y_1] + v[y_1'' + a(x)y_1' + b(x)y_1] = 0,
\]

Using the fact that \( y_1 \) is a solution to eq. (8), the coefficient of \( v \) vanishes and we are left with a first order differential equation for \( w \)

\[
w'y_1 + w[2y_1' + a(x)y_1] = 0.
\]

After dividing this equation by \( y_1' \), we see that the solution to the resulting equation is

\[
w(x) = c \exp \left\{ - \int \left( \frac{2y_1'(x)}{y_1(x)} + a(x) \right) dx \right\}
\]

\[
= ce^{-2\ln y_1(x)} \exp \left\{ - \int a(x)dx \right\}
\]

\[
= \frac{c}{[y_1(x)]^2} \exp \left\{ - \int a(x)dx \right\} = \frac{W(x)}{[y_1(x)]^2},
\]

*More generally, if any non-trivial solution to eq. (3) is known, then this solution can be employed to reduce the order of the differential equation by 1. This procedure is called reduction of order.
after using eq. (9) for the Wronskian. The second solution to eq. (8) defined by eq. (10) is then given by

\[ y_2(x) = y_1(x) \int \frac{W(x)}{[y_1(x)]^2} \, dx \]

after employing eq. (12).

Finally, we note that the Wronskian also appears in solutions to inhomogeneous linear differential equations. For example, consider

\[ y'' + a(x)y' + b(x)y = f(x), \quad (13) \]

and assume that the solutions to the homogeneous equation [eq. (8)], denoted by \( y_1(x) \) and \( y_2(x) \) are known. Then the general solution to eq. (13) is given by

\[ y(x) = c_1y_1(x) + c_2y_2(x) + y_p(x), \]

where \( y_p(x) \), called the particular solution, is determined by the following formula,

\[ y_p(x) = -y_1(x) \int \frac{y_2(x)f(x)}{W(x)} \, dx + y_2(x) \int \frac{y_1(x)f(x)}{W(x)} \, dx. \quad (14) \]

This result is derived using the technique of variation of parameters. Namely, one writes

\[ y_p(x) = v_1(x)y_1(x) + v_2(x)y_2(x), \quad (15) \]

subject to the condition (which is chosen entirely for convenience):

\[ v_1'y_1 + v_2'y_2 = 0. \quad (16) \]

With this choice, it follows that

\[ y_p' = v_1'y_1 + v_2'y_2. \]

Differentiating once more and plugging back into eq. (13), one obtains [after using eq. (16)]:

\[ v_1'y_1' + v_2'y_2' = f(x). \quad (17) \]

We now have two equations, eqs. (16) and (17), which constitute two algebraic equations for \( v_1' \) and \( v_2' \). The solutions to these equations yield

\[ v_1' = -\frac{y_2(x)f(x)}{W(x)}, \quad v_2' = \frac{y_1(x)f(x)}{W(x)}, \]

where \( W(x) \) is the Wronskian. We now integrate to get \( v_1 \) and \( v_2 \) and plug back into eq. (15) to obtain eq. (14). The derivation is complete.

**Reference:**

APPENDIX I: Derivative of the determinant of a matrix

Recall that for any matrix $A$, the determinant can be computed by the cofactor expansion. The adjugate of $A$, denoted by $\text{adj } A$ is equal to the transpose of the matrix of cofactors. In particular,

$$\det A = \sum_j a_{ij}(\text{adj } A)_{ji}, \quad \text{for any fixed } i,$$  \hspace{2cm} (18)

where the $a_{ij}$ are elements of the matrix $A$ and $(\text{adj } A)_{ji} = (-1)^{i+j}M_{ij}$ where the minor $M_{ij}$ is the determinant of the matrix obtained by deleting the $i$th row and $j$th column of $A$.

Suppose that the elements $a_{ij}$ depend on a variable $x$. Then, by the chain rule,

$$\frac{d}{dx} \det A = \sum_{i,j} \frac{\partial \det A}{\partial a_{ij}} \frac{da_{ij}}{dx}.$$  \hspace{2cm} (19)

Using eq. (18), and noting that $(\text{adj } A)_{ji}$ does not depend on $a_{ij}$ (since the $i$th row and $j$th column are removed before computing the minor determinant),

$$\frac{\partial \det A}{\partial a_{ij}} = (\text{adj } A)_{ji}.$$  

Hence, eq. (19) yields Jacobi’s formula:†

$$\frac{d}{dx} \det A = \sum_{i,j} (\text{adj } A)_{ji} \frac{da_{ij}}{dx} = \text{Tr} \left[ (\text{adj } A) \frac{dA}{dx} \right].$$  \hspace{2cm} (20)

If $A$ is invertible, then we can use the formula

$$A^{-1} \det A = \text{Adj } A,$$

to rewrite eq. (20) as‡

$$\frac{d}{dx} \det A = \det A \text{Tr} \left( A^{-1} \frac{dA}{dx} \right),$$  \hspace{2cm} (21)

which is the desired result.

Reference:


†Recall that if $A = [a_{ij}]$ and $B = [b_{ij}]$, then the $ij$ matrix element of $AB$ are given by $\sum_k a_{ik}b_{kj}$. The trace of $AB$ is equal to the sum of its diagonal elements, or equivalently

$$\text{Tr}(AB) = \sum_{jk} a_{jk}b_{kj}.$$  

‡Note that $\text{Tr } (cB) = c \text{Tr } B$ for any number $c$ and matrix $B$. In deriving eq. (21), $c = \det A$. 

5
APPENDIX II: Derivation of Abel’s equation

Eq. (9) is usually called Abel’s formula, and can be derived directly without using any fancy techniques. Consider the ordinary linear second order differential equation,
\[ y'' + a(x)y' + b(x)y = 0. \tag{22} \]
Suppose that \( y_1(x) \) and \( y_2(x) \) are linearly independent solutions of eq. (22). Then the Wronskian is non-vanishing,
\[ W = \det \begin{pmatrix} y_1 & y_2 \\ y'_1 & y'_2 \end{pmatrix} = y_1y'_2 - y'_1y_2 \neq 0. \]
Taking the derivative of the above equation,
\[ \frac{dW}{dx} = \frac{d}{dx} (y_1y'_2 - y'_1y_2) = y_1y''_2 - y''_1y_2, \]
since the terms proportional to \( y'_1y'_2 \) exactly cancel. Using the fact that \( y_1 \) and \( y_2 \) are solutions to eq. (22), we have
\[ y''_1 + a(x)y'_1 + b(x)y_1 = 0, \tag{23} \]
\[ y''_2 + a(x)y'_2 + b(x)y_2 = 0. \tag{24} \]
Next, we multiply eq. (24) by \( y_1 \) and multiply eq. (23) by \( y_2 \), and subtract the resulting equations. The end result is:
\[ y_1y''_2 - y''_1y_2 + a(x) [y_1y'_2 - y'_1y_2] = 0. \]
or equivalently [cf. eq. (6)],
\[ \frac{dW}{dx} + a(x)W = 0, \tag{25} \]
The solution to this first order differential equation is Abel’s formula,
\[ W(x) = c \exp \left\{ - \int a(x)dx \right\}, \tag{26} \]
where \( c \) is an arbitrary constant. Eq. (26) is a special case of Liouville’s formula given in eq. (7) for ordinary second-order linear differential equations \( (n = 2) \).