1 Boas, p. 632, problem 13.3-2

A bar 10 cm long with insulated sides is initially at 100$^{\text{o}}$. Starting at $t = 0$, the ends are held at 0$^{\text{o}}$. Find the temperature distribution in the bar at time t.

The heat flow equation is

$$\nabla^2 u = \frac{1}{\alpha^2}\frac{\partial u}{\partial t} \tag{1}$$

where $u(x, t)$ is the temperature. Because the sides of the bar are insulated, the heat flows only in the x direction; the same happens for a slab of finite thickness but infinitely large. The initial condition is $u(x, 0) = 100$ and the boundary condition for $t > 0$ is $u(0, t) = u(10, t) = 0$.

We search for a solution of the form $u(x, t) = F(x)T(t)$; the differential equation (1) gives

$$\frac{\partial^2 F}{\partial x^2} + k^2 F = 0\,, \qquad \Longrightarrow \qquad F = A\cos kx + B\sin kx, \tag{2}$$

$$\frac{\partial T}{\partial t} = -k^2\alpha^2 T\,, \qquad \Longrightarrow \qquad T = e^{-k^2\alpha^2 t}. \tag{3}$$

Because of the boundary condition in $x = 0$, we have $A = 0$. To satisfy $u(10, t) = 0$, we find discrete values of k:

$$\sin 10k = 0\,, \Longrightarrow\ k_n = \frac{n\pi}{10}, \quad \text{for } n = 1, 2, 3, \ldots \tag{4}$$

Finally, we must satisfy the initial condition

$$u(x, 0) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{10} = 100\,, \quad \text{for } 0 \le x \le 10. \tag{5}$$

Solving for the b_n using eq. (2.11) on p. 623 of Boas,

$$b_n = \frac{2}{10}\int_0^{10} 100\sin\frac{n\pi x}{10}dx = -\frac{200}{n\pi}\cos\frac{n\pi x}{10}\bigg|_0^{10} \tag{6}$$

$$= \frac{200}{n\pi}\left[1 - (-1)^n\right] = \begin{cases} \dfrac{400}{n\pi}\,, & \text{for odd } n\,, \\ 0\,, & \text{for even } n\,, \end{cases} \tag{7}$$

after using $\cos n\pi = (-1)^n$ for integer n. The solution to the differential equation (1) with the given boundary conditions is then

$$u = \frac{400}{\pi}\sum_{\text{odd } n}\frac{1}{n}e^{-(n\pi\alpha/10)^2 t}\sin\frac{n\pi x}{10}, \tag{8}$$

where "odd n" means a sum over $n = 1, 3, 5, \ldots$.

2 Boas, p. 632, problem 13.3-6

Show that the following problem is easily solved using eq. (3.15) in Boas, p. 630: the ends of a bar are initially at 20° and 150°; at $t = 0$ the 150° end is changed to 50°. Find the time-dependent temperature distribution.

The suggested equation is

$$u = \frac{200}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}e^{-(n\pi\alpha/l)^2 t}\sin\frac{n\pi x}{l}. \tag{9}$$

and describes a bar of length l with ends at initial temperatures 0° and 100°, which starting at $t = 0$ has both ends at 0°.

The initial temperature distribution for our problem is $u(x, 0) = 20 + \frac{130x}{l}$, while the boundary conditions for $t > 0$ are $u(0, t) = 20$, $u(l, t) = 50$. Because the heat flow equation is linear and the constant distribution

$$u_0 = 20 + 30\frac{x}{l} \tag{10}$$

satisfies it with the boundary conditions $u(x, 0) = u_0$, $u(0, t) = 0$, $u(l, t) = 50$, the function $u + u_0$ satisfies the heat equation with the boundary conditions required by this problem and is the solution we were looking for.

3 Boas, p. 633, problem 13.3-8

A bar of length 2 is initially at 0°. From $t = 0$ on , the $x = 0$ end is held at 0° and the $x = 2$ end at 100°. Find the time dependent-temperature distribution.

The boundary conditions are

$$u(x, 0) = 0, \qquad u(0, t) = 0, \qquad u(2, t) = 100. \tag{11}$$

As in the last problem we can use the linearity of the heat flow equation: a time-independent $u(x, t) = 50x$ satisfies (1) with boundary conditions $u(x, 0) = 50x\,, u(0, t) = 0\,, u(2, t) = 100$. If we subtract (9) to this one, we get the solution to our problem

$$u(x, t) = 50x - \frac{200}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}e^{-(n\pi\alpha/2)^2 t}\sin\frac{n\pi x}{2}, \tag{12}$$

after setting the length $l = 2$.

4 Boas, p. 633, problem 13.3-12

Solve the "particle in a box" problem to find $\Psi(x, t)$ if $\Psi(x, 0) = \sin^2 \pi x$ in $(0, 1)$. What is E_n?

Schrödinger equation reads

$$-\frac{\hbar^2}{2m}\nabla^2\Psi = i\hbar\frac{\partial}{\partial t}\Psi \tag{13}$$

If we separate the variables, $\Psi(x, t) = \psi(x)T(t)$, this becomes

$$-\frac{\hbar^2}{2m}\frac{1}{\psi}\nabla^2\psi = i\hbar\frac{1}{T}\frac{dT}{dt} = E, \qquad \Longrightarrow \qquad T(t) = e^{-iEt/\hbar}\,, \qquad \psi = A\cos kx + B\sin kx \tag{14}$$

where $k^2 = \frac{2mE}{\hbar^2}$. In the "particle in a box" scenario, the wave function is zero at the boundaries

$$\psi(0) = 0 \qquad \psi(l) = 0 \tag{15}$$

and in our case we have the initial condition $\psi(x) = \sin^2 \pi x$. The boundary conditions give $A = 0$ and $k_n = \frac{n\pi}{l} = n\pi$, thus implying $E_n = \frac{\hbar^2 n^2 \pi^2}{2m}$. We must satisfy the initial condition

$$\Psi(x,0) = \sum_n B_n \sin n\pi x = \sin^2 \pi x \tag{16}$$

$$B_n = \frac{2}{l}\int_0^l \sin^2 \pi x \sin \frac{n\pi x}{l} dx = \frac{2}{\pi}\int_0^\pi \sin^2 y \sin ny\, dy = \tag{17}$$

$$= \frac{1}{\pi}\int_0^\pi (1 - \cos 2y)\sin ny dy = \frac{1}{\pi}\left[-\frac{1}{n}[(-1)^n - 1] - \frac{1}{2}\int_0^\pi dy(\sin(2+n)y - \sin(2-n)y)\right] \tag{18}$$

$$= \frac{1}{\pi}\left[\frac{1}{n} - \frac{1}{2}\left(\frac{1}{2+n} - \frac{1}{2-n}\right)\right][-(-1)^n + 1] = \begin{cases} 0 & \text{for even } n; \\ \frac{8}{\pi}\frac{1}{n(4-n^2)} & \text{for odd } n. \end{cases} \tag{19}$$

The wave function and the relative energy eigenvalues are

$$E_n = \frac{\hbar^2 n^2 \pi^2}{2m}, \qquad \Psi(x,t) = \frac{8}{\pi}\sum_{odd\ n} \frac{\sin n\pi x}{n(4-n^2)} e^{-iE_n t/\hbar} \tag{20}$$

5 Boas, p. 637, problem 13.4-4

A string of length l has zero initial velocity and a given displacement $y_0(x)$. Find the displacement as a function of x and t.

$$y_0(x) = \begin{cases} \frac{4h}{l}x, & 0 < x < \frac{l}{4}, \\ 2h - \frac{4h}{l}x, & \frac{l}{4} < x < \frac{3}{4}l, \\ -4h + \frac{4h}{l}x, & \frac{3}{4}l < x < l; \end{cases} \tag{21}$$

The wave equation is

$$\frac{\partial^2 y}{\partial^2 x} = \frac{1}{v^2}\frac{\partial^2 y}{\partial^2 t}; \tag{22}$$

separating the variables $y(x,t) = X(x)T(t)$, the solution is

$$y(x,t) = \left\{ \begin{array}{c} \sin kx \\ \cos kx \end{array} \right\} \times \left\{ \begin{array}{c} \sin \omega t \\ \cos \omega t \end{array} \right\}, \qquad \omega = kv. \tag{23}$$

Now we must apply the given boundary and initial conditions. The string is fixed at its ends so that $\cos kx$ does not contribute; because $y(l,t) = 0$, we have $k_n = \frac{n\pi}{l}$. The solution can be written as:

$$y(x,t) = \sum_n \sin k_n x\,(A_n \cos \omega_n t + B_n \sin \omega_n t) \tag{24}$$

Because the string is initially not moving we have $\frac{\partial y}{\partial t}(x,0) = 0$, implying $B_n = 0$. Finally, we find the coefficients A_n from the initial configuration of the string:

$$y(x,0) = \sum_n A_n \sin k_n x = y_0(x); \tag{25}$$

$$\begin{aligned} A_n &= \frac{2}{l}\int_0^l y_0(x)\sin k_n x\, dx = \frac{2}{l}\int_0^{l/4} dx\, \frac{4h}{l}x \sin\frac{n\pi}{l}x + \frac{2}{l}\int_{l/4}^{3l/4} dx\, \frac{4h}{l}(-x+\tfrac{l}{2})\sin\frac{n\pi}{l}x + \\ &\quad + \frac{2}{l}\int_{3l/4}^{l} dx\, \frac{4h}{l}(x-l)\sin\frac{n\pi}{l}x = \end{aligned} \tag{26}$$

$$\begin{aligned} &= \frac{8h}{l^2}\left\{\left[-x\frac{l}{n\pi}\cos\frac{n\pi}{l}x\Big|_0^{l/4} + \frac{l^2}{n^2\pi^2}\sin\frac{n\pi}{l}x\Big|_0^{l/4}\right] + \left[-(-x+\tfrac{l}{2})\frac{l}{n\pi}\cos\frac{n\pi}{l}x\Big|_{l/4}^{3l/4} - \frac{l^2}{n^2\pi^2}\sin\frac{n\pi}{l}x\Big|_{l/4}^{3l/4}\right]\right. \\ &\quad \left. + \left[-(x-l)\frac{l}{n\pi}\cos\frac{n\pi}{l}x\Big|_{3l/4}^{l} + \frac{l^2}{n^2\pi^2}\sin\frac{n\pi}{l}x\Big|_{3l/4}^{l}\right]\right\} = \frac{32h}{n^2\pi^2}\cos\frac{n\pi}{2}\sin\frac{n\pi}{4} \end{aligned} \tag{27}$$

For odd n, $\cos n\pi/2 = 0$, so that only the even n's contribute. The solution to the wave equation with the given initial condition is

$$y = \frac{8h}{\pi^2}\sum_n \frac{1}{n^2}\sin\frac{n\pi}{2}\sin\frac{2n\pi x}{l}\cos\frac{2n\pi vt}{l} \tag{28}$$

6 Boas, p. 638, problem 13.4-7

A string of length l is initially stretched straight; its ends are fixed for all t. At the time $t = 0$ its points are given a velocity $V(x)$. Determine the shape of the string at a time t.

$$V(x) = \begin{cases} h\frac{3}{l}x, & 0 < x < \frac{l}{3}, \\ h\frac{3}{2l}(l-x) & \frac{l}{3} < x < l; \end{cases} \tag{29}$$

The problem is the same as in the last exercise, except that we have different initial conditions: instead of the initial position $y(x,0)$ we have the initial velocity $\frac{\partial y}{\partial t}(x,0)$. As before the solutions to the equation (22) are given by (23). Because the ends are fixed, $\cos kx$ does not contribute and because $y(l,t) = 0$, we have $k_n = \frac{n\pi}{l}$. The solution can be written as:

$$y(x,t) = \sum_n \sin k_n x\,(A_n\cos\omega_n t + B_n\sin\omega_n t) \tag{30}$$

This time, because $y(x,0) = 0$, we have $A_n = 0$; we find the B_n's applying the initial condition:

$$\frac{\partial y}{\partial t}(x,0) = \sum_n B_n\frac{n\pi v}{l}\sin k_n x = V(x); \tag{31}$$

$$\frac{n\pi v}{l}B_n = \frac{2}{l}\int_0^l V(x)\sin k_n x\, dx = \frac{2}{l}\int_0^{l/3} dx\frac{3h}{l}x\sin\frac{n\pi x}{l} + \frac{2}{l}\int_{l/3}^{l} dx\frac{3h}{2l}(l-x)\sin\frac{n\pi x}{l} \tag{32}$$

$$\begin{aligned} &= \frac{6h}{l^2}\left\{\left[-\frac{l}{n\pi}x\cos\frac{n\pi x}{l}\Big|_0^{l/3} + \frac{l^2}{n^2\pi^2}\sin\frac{n\pi x}{l}\Big|_0^{l/3}\right] + \frac{1}{2}\left[-\frac{l}{n\pi}(l-x)\cos\frac{n\pi x}{l}\Big|_{l/3}^{l} - \frac{l^2}{n^2\pi^2}\sin\frac{n\pi x}{l}\Big|_{l/3}^{l}\right]\right\} \\ &= \frac{9h}{n^2\pi^2}\sin\frac{n\pi}{3} \qquad\qquad B_n = \frac{9hl}{n^3\pi^3 v}\sin\frac{n\pi}{3} \end{aligned} \tag{33}$$

The solution with these initial conditions is then

$$y = \frac{9hl}{\pi^3 v}\sum_n \frac{1}{n^3}\sin\frac{n\pi}{3}\sin\frac{n\pi x}{l}\sin\frac{n\pi vt}{l}. \tag{34}$$

7 Boas, p. 638, problem 13.4-9

Find the frequency of the most important harmonic for the previous two problems.

In problem 5 the frequencies are

$$f_n = \frac{\omega_n}{2\pi} = \frac{nv}{l}\,,\ n = 1, 2, 3, \ldots \tag{35}$$

Due to the factor of $1/n^2$ in the Fourier expansion, the most important harmonic is the fundamental frequency, which is $f_1 = v/l$.

In problem 6 the frequencies are

$$f_n = \frac{\omega_n}{2\pi} = \frac{\pi v}{2l}\,,\ n = 1, 2, 3, \ldots \tag{36}$$

Due to the factor of $1/n^3$ in the Fourier expansion, the most important harmonic is the fundamental frequency, which is $f_1 = v/(2l)$.

8 Boas, p. 646, problem 13.6-2

Find the first three zeros k_{mn} of each of the Bessel functions J_0, J_1, J_2, J_3. Find the first six frequencies of a vibrating circular membrane as (non-integral) multiples of the fundamental frequency.

We denote by k_{mn} the set of the zeros of the Bessel function J_n; then we have[1]

$$k_{mn}\big|_{n=0,1,2,3} = \begin{array}{c|cccc} m & J_0 & J_1 & J_2 & J_3 \\ \hline 1 & 2.4048 & 3.8317 & 5.1356 & 6.3802 \\ 2 & 5.5201 & 7.0156 & 8.4172 & 9.7610 \\ 3 & 8.6537 & 10.1735 & 11.6198 & 13.0152 \end{array} \tag{37}$$

The frequencies of the fundamental modes of a circular membrane of radius a are given by

$$\nu_{mn} = \frac{k_{mn} v}{2\pi a} \tag{38}$$

We can write the first six frequencies in terms of $\nu_0 = \nu_{10}$:

$$v_1 = \frac{k_{11}}{k_{10}}\nu_0 = 1.593\nu_0\,, \qquad v_2 = \frac{k_{12}}{k_{10}}\nu_0 = 2.136\nu_0\,, \qquad v_3 = \frac{k_{20}}{k_{10}}\nu_0 = 2.295\nu_0\,, \tag{39}$$

$$v_4 = \frac{k_{13}}{k_{10}}\nu_0 = 2.653\nu_0\,, \qquad v_5 = \frac{k_{21}}{k_{10}}\nu_0 = 2.917\nu_0\,, \qquad v_6 = \frac{k_{22}}{k_{10}}\nu_0 = 3.5\nu_0. \tag{40}$$

9 Boas, p. 646–647, problem 13.6-3

Separate the wave equation in two-dimensional rectangular coordinates x, y. Consider a rectangular membrane rigidly attached to supports along its sides. Show that the characteristic frequencies are

$$n_{nm} = \tfrac{v}{2}\sqrt{(n/a)^2 + (m/b)^2}, \tag{41}$$

where n, m are positive integers, and sketch the normal modes of vibration corresponding to the first few frequencies

[1] In `Mathematica`, they can be found with the command `BesselJZero[n, m]`. One can also find convenient tables of Bessel function zeros at `http://mathworld.wolfram.com/BesselFunctionZeros.html`.

Next, suppose the membrane is square. Show that there may be several normal modes of vibration corresponding to one frequency (that is, they are *degenerate*).

The wave equation in rectangular coordinates is

$$\nabla^2 z(x,y,t) = \frac{1}{v^2}\frac{\partial^2 z}{\partial t^2} \tag{42}$$

If we first separate the space and time variables, $z(x,y,t) = F(x,y)T(t)$, we have

$$\nabla^2 F + K^2 F = 0 \qquad\qquad \ddot{T} + K^2 v^2 T = 0 \tag{43}$$

The equation for $T(t)$ has sines and cosines as solutions. If we separate the equation for F in rectangular coordinates, $F(x,y) = X(x)Y(y)$, we have $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ and

$$\frac{X''(x)}{X} + \frac{Y''(y)}{Y} + K^2 = 0 \qquad \Longrightarrow \qquad \begin{cases} X'' + k_x^2 X = 0 \\ Y'' + k_y^2 Y = 0 \end{cases}, \; k_x^2 + k_y^2 = K^2 \tag{44}$$

The solutions for both X and Y are a combination of sines and cosines; given that the sides of the rectangular membrane are fixed, the boundary conditions are

$$z(0,y,t) = z(a,y,t) = z(x,0,t) = z(x,b,t) = 0 \tag{45}$$

For X, this tells that

$$X(x) = \sin k_n x, \; k_n = \frac{n\pi}{a}. \tag{46}$$

and the same happens for $Y(y)$. The full solution will then be

$$Z(x,y,t) = \sum_{nm} \sin\frac{n\pi x}{a}\, \sin\frac{m\pi y}{b}\left(A_{nm}\cos\omega_{nm}t + B_{nm}\sin\omega_{nm}t\right), \qquad \omega_{nm} = K_{nm}v, \tag{47}$$

where $K_{nm} = \sqrt{k_{x(n)}^2 + k_{y(m)}^2} = \pi\sqrt{(n/a)^2 + (m/b)^2}$. The coefficients A, B will be found given the initial configuration of the membrane (for example, if the membrane is initially flat, we will require $A = 0$, while if it is initially not moving we will have $B = 0$). The frequencies of the oscillation are given by

$$\nu_{nm} = \frac{\omega_{nm}}{2\pi} = \frac{v}{2}\sqrt{(n/a)^2 + (m/b)^2}. \tag{48}$$

For a square membrane, this becomes

$$\nu_{nm} = \frac{v}{2a}\sqrt{n^2 + m^2}; \tag{49}$$

Then one can see that it is possible for different values of (n,m) to give the same values of ν_{nm}. This happens every time we have two couples of integers (n,m) and (n',m') satisfying $n^2 + m^2 = n'^2 + m'^2$. For example, this happens for

$$(n,m) = (1,2), \qquad (n',m') = (2,1) \implies \nu = \sqrt{3}\frac{v}{2a}; \tag{50}$$

$$(n,m) = (1,7), \qquad (n',m') = (5,5), \qquad (n'',m'') = (7,1) \implies \nu = \sqrt{50}\frac{v}{2a} \tag{51}$$

In these cases, we say that the different modes are *degenerate*.

10 Boas, p. 647, problem 13.6-8

Let $V = 0$ in the Schrödinger equation and separate variables in polar coordinates to find the eigenfunctions and energy eigenvalues of a particle in a circular box $r < a$. On the boundary, $\Psi(r = a) = 0$.

We want to solve the Schrödinger equation in a circular box:

$$-\frac{\hbar^2}{2M}\nabla^2\Psi = i\hbar\frac{\partial}{\partial t}\Psi\,, \tag{52}$$

where M is the mass of the particle in the box. As usual, we separate the time coordinate:

$$\Psi(x,t) = \psi(x)T(t) \implies \begin{cases} T(t) = T(0)e^{-iEt/\hbar} \\ -\dfrac{\hbar^2}{2M}\nabla^2\psi = E\psi \end{cases} \tag{53}$$

The eigenvalue equation can be solved in spherical coordinates $\psi(x,y) = R(r)\Theta(\theta)$, while the Laplacian in polar coordinates is

$$\nabla^2 = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2} = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2} \tag{54}$$

In is convenient to introduce the positive wave number k,

$$k \equiv \sqrt{\frac{2ME}{\hbar^2}}\,, \tag{55}$$

in which case $\psi(x)$ satisfies the eigenvalue equation,

$$\nabla^2\psi(x) = -k^2\psi(x)\,. \tag{56}$$

Using (54),

$$R''\Theta + \frac{\Theta}{r}R' + \frac{R}{r^2}\Theta'' = -k^2R\Theta \qquad \implies \qquad r^2\frac{R''}{R} + r\frac{R'}{R} + r^2k^2 = -\frac{\Theta''}{\Theta} = +n^2. \tag{57}$$

The angular equation has the solution $\Theta(\theta) = \{\sin n\theta\,, \cos n\theta\}$. For the solution to be single-valued, that is, satisfying $\Theta(\theta + 2\pi) = \Theta(\theta)$, we require that n is a non-negative integer; the radial equation becomes

$$r^2R'' + rR' + (r^2k^2 - n^2)R = 0 \tag{58}$$

This is Bessel equation and we can express it in its standard form by changing variable, $x = kr$; then we have

$$x^2R'' + xR' + (x^2 - n^2)R = 0 \tag{59}$$

R will then be a Bessel function, $R(r) = J_n(kr)$. The full solution for the spatial wave function is

$$\psi(r,\theta) = J_n(kr)\left\{\begin{array}{c}\sin n\theta \\ \cos n\theta\end{array}\right\}, \qquad n = 0, 1, 2, \ldots \tag{60}$$

Now we only have to impose the boundary condition $R(r = a) = 0$, that is $J_n(ka) = 0$. This is satisfied whenever ka is a positive zero of the Bessel function J_n. If we label the m-th zero of $J_n(x)$ by x_{mn}, we have

$$x_{mn} = k_{mn}a \tag{61}$$

This set the allowed values of the energy and solves the eigenvalue problem. The solution for the full problem is

$$\Psi_{nm} = J_n(k_{mn}r)\left\{\begin{array}{c}\sin n\theta \\ \cos n\theta\end{array}\right\}e^{-iE_{mn}t/\hbar}, \qquad E_{mn} = \frac{\hbar^2k_{mn}^2}{2M}, \tag{62}$$

after using (55) to obtain the allowed values of the energy eigenvalues E.