1 Boas, p. 643, problem 13.5-3(b)

Find the steady-state temperature distribution in a solid cylinder of height H and radius a if the top and curved surfaces are held at 0 and the base at 100.

We have to solve the equation

$$\nabla^2 u(r,\theta,z) = 0 \tag{1}$$

subject to the boundary conditions

$$u(r,\theta,H) = u(a,\theta,z) = 0\,, \qquad u(r,\theta,0) = 100 \tag{2}$$

After the usual separation of variables in cylindrical coordinates $u(r,\theta,z) = R(r)\Theta(\theta)Z(z)$ we are left with

$$\frac{1}{rR}(rR')' + \frac{1}{r^2\Theta}\Theta'' = -\frac{Z''}{Z} = -K^2 \tag{3}$$

The $R-\Theta$ equation can also be separated and gives

$$\frac{r}{R}(rR')' + K^2 r^2 = -\frac{\Theta''}{\Theta} = n^2 \qquad \Longrightarrow \qquad \begin{cases} \Theta(\theta) = e^{\pm in\theta}, \\ R(r) = J_n(Kr) \end{cases} \tag{4}$$

n has to be an integer so that $\Theta(\theta+2\pi) = \Theta(\theta)$. The Z equation gives $Z(z) = A\sinh Kz + B\cosh Kz = A'\sinh K(H-z)$, were we have parametrized the solution so that Z(H)=0. The full solution is

$$u(r,\theta,z) = J_n(Kr)\sinh K(H-z)e^{\pm in\theta} \tag{5}$$

We now have to impose the boundary conditions (2), but first we notice that they have a rotational symmetry around the z axis (that is, there is no θ-dependence): then there will be no θ-dependence in the answer, that is, we have $n=0$. For the other boundary conditions,

$$u(a,\theta,z) = 0 \implies J_0(Ka) = 0 \implies Ka = x_m \text{ is a zero of the Bessel function } J_0(x) \tag{6}$$

$$u = \sum_m c_m J_0(x_m r/a)\sinh[x_m(H-z)/a] \tag{7}$$

$$u(r,\theta,0) = 100 = \sum_m c_m J_0(x_m r/a)\sinh(x_m H/a) \tag{8}$$

Because the set $\{J_0(x_m r/a)\,, m = 1,2,\ldots\}$ is an orthogonal set in $(0,a)$, we can find the coefficient c_n by multiplying expression (8) by $J_0(x_n r/a)$ and integrating between 0 and a. Then we get (following Boas, p.641)

$$\sinh(x_n H/a)c_n = \frac{200}{x_n J_1(x_n)} \tag{9}$$

or

$$u(r,\theta,z) = \sum_{m=1}^{\infty} \frac{200}{x_m J_1(x_m)\sinh(x_m H/a)} J_0(x_m r/a)\sinh[x_m(H-z)/a]. \tag{10}$$

2 Boas, p. 643, problem 13.5-5

A flat circular plate of radius a has a temperature distribution $u(r,\theta) = 100r\sin\theta$. At $t = 0$, the circumference of the plate is held at 0. Find the time dependent temperature distribution.

We have to solve the problem

$$\nabla^2 u = \frac{1}{\alpha^2}\frac{\partial}{\partial t}u\,, \qquad \left\{ \begin{array}{l} u(r,\theta,0) = 100r\sin\theta, \\ u(a,\theta,t) = 0,\ t > 0 \end{array} \right. \tag{11}$$

We solve the differential equation in the usual way:

$$u = F(x,y)T(t) \implies \frac{1}{F}\nabla^2 F = \frac{1}{\alpha^2}\frac{T''}{T} = -k^2 \tag{12}$$

The solution for T is $T(t) = e^{-k^2\alpha^2 t}$, while for F we have

$$F(x,y) = R(r)\Theta(\theta) \implies \frac{r}{R}(rR')' + k^2r^2 = -\frac{\Theta''}{\Theta} = n^2 \implies \left\{ \begin{array}{l} \Theta(\theta) = e^{\pm in\theta}, \\ R(r) = J_n(kr) \end{array} \right. \tag{13}$$

The boundary condition $R(a) = 0$ tells us that ka is a zero of the n-th Bessel function; if we let x_{mn} be the m-th zero of the Bessel function J_n, then $ka = x_{mn}$. Unlike in the last problem, we now have a θ-dependence in the initial condition:

$$u(r,\theta,0) = 100r\sin\theta = \sum_{m,n} J_n(x_{mn}r/a)(A_{mn}\cos n\theta + B_{mn}\sin n\theta) \tag{14}$$

By inspection, we already see there will be no $\cos n\theta$ dependence on the right hand side ($A_{mn} = 0$). If we multiply the previous equation by $\sin l\theta$ and integrate over $\theta \in (0, 2\pi)$, we have

$$100r\int_0^{2\pi} \sin l\theta \sin\theta\, d\theta = 100J_1(r)\delta_{l1}\pi = \sum_{n,m} J_n(x_{mn}r/a)B_{ml}\pi\delta_{ln}\ , \tag{15}$$

$$\implies B_{ml} = 0 \text{ for } l \neq 1, \qquad 100r = \sum_m J_1(x_{m1}r/a)B_{m1}\,. \tag{16}$$

If we now multiply by $rJ_1(x_{k1}r/a)$ and integrate over $r \in (0,a)$, we get

$$100\int_0^a r^2 J_1(x_{k1}r/a)dr = 100\int_0^a \frac{ar}{x_{k1}}\frac{d}{dr}rJ_2(x_{k1}r/a)dr = 100\frac{a^3}{x_{k1}}J_2(x_{k1}) = \tag{17}$$

$$= \sum_m B_{m1}\int_0^a rJ_1(x_{m1}r/a)J_1(x_{k1}r/a)dr = B_{k1}\frac{a^2}{2}J_2^2(x_{k1}) \tag{18}$$

Hence,

$$B_{k1} = \frac{200a}{x_{k1}J_2(x_{k1})}\,. \tag{19}$$

So that the final solution is

$$u = \sum_{k=1}^{\infty} \frac{200a}{x_{k1}J_2(x_{k1})} J_1(x_{k1}r/a)e^{-x_{k1}^2\alpha^2/a^2 t} \qquad \text{with } x_{k1}\text{=zeros of } J_1(x). \tag{20}$$

3 Boas, p. 643, problem 13.5-7

Find the steady state temperature distribution in a solid cylinder of height 20 and radius 3 if the flat ends are held at 0 and the curved surface is at 100.

We have to solve the problem

$$\nabla^2 u(r,\theta,z) = 0\,, \qquad \begin{cases} u(r,\theta,0) = u(r,\theta,20) = 0, \\ u(a,\theta,z) = 100. \end{cases} \tag{21}$$

Following the derivation on pp. 639–640 of Boas, we write

$$u(r,\theta,z) = R(r)\Theta(\theta)Z(z)\,, \tag{22}$$

and obtain

$$\frac{1}{R}\frac{1}{r}\frac{d}{dr}\left(r\frac{dR}{dr}\right) + \frac{1}{\Theta}\frac{1}{r^2}\frac{d^2\Theta}{d\theta^2} + \frac{1}{Z}\frac{d^2Z}{dz^2} = 0\,. \tag{23}$$

Choosing the opposite sign for the separation constant as compared to eq. (5.4) on p. 639 of Boas, we have

$$\frac{1}{Z}\frac{d^2Z}{dz^2} = -k^2\,, \qquad Z = \begin{cases} \sin kz\,, \\ \cos kz\,. \end{cases} \tag{24}$$

Inserting this result back into (23),

$$\frac{1}{R}\frac{1}{r}\frac{d}{dr}\left(r\frac{dR}{dr}\right) + \frac{1}{\Theta}\frac{1}{r^2}\frac{d^2\Theta}{d\theta^2} - k^2 = 0\,. \tag{25}$$

We can separate the variables by multiplying through by r^2,

$$\frac{r}{R}\frac{d}{dr}\left(r\frac{dR}{dr}\right) + \frac{1}{\Theta}\frac{d^2\Theta}{d\theta^2} - k^2r^2 = 0\,. \tag{26}$$

Thus we have,

$$\frac{1}{\Theta}\frac{d^2\Theta}{d\theta^2} = -n^2\,, \qquad \Theta = \begin{cases} \sin n\theta\,, \\ \cos n\theta\,. \end{cases} \tag{27}$$

Inserting this back into (26), we obtain the radial equation,

$$\frac{r}{R}\frac{d}{dr}\left(r\frac{dR}{dr}\right) - n^2 - k^2r^2 = 0\,, \tag{28}$$

or

$$r^2\frac{d^2R}{dr^2} + r\frac{dR}{dr} - (k^2r^2 + n^2)R = 0\,. \tag{29}$$

Comparing this differential equation with eq. (17.2) on p. 595 of Boas, we identify the possible solutions as:

$$R(r) = \begin{cases} I_n(kr)\,, \\ K_n(kr)\,. \end{cases} \tag{30}$$

However, since the present problem includes the origin, we demand that the radial solution should be finite as $r \to 0$. In light of the small r behavior of $I_n(kr)$ and $K_n(kr)$ as specified on p. 604 of Boas, we must exclude $K_n(kr)$ as a possible solution. Hence,

$$R(r) = I_n(kR)\,. \tag{31}$$

If we now impose the requirement that the temperature is held at $0°$ at $z = 0$ and $100°$ on the surface of the solid cylinder, then it follows that $n = 0$ since the boundary value temperature has no θ-dependence. Hence, the allowed solutions for the temperature are given by:

$$u(r, \theta, z) = I_0(kr) \sin kz \,. \tag{32}$$

Next, we impose the boundary condition that the temperature is held at $0°$ at $z = 20$. This implies that $20k = n\pi$ or $k = m\pi/20$, for $m = 1, 2, 3, \ldots$. Hence, the solution must be of the form,

$$u(r, \theta, z) = \sum_{m=1}^{\infty} c_m I_0(\pi m r/20) \sin(\pi m z/20) \,. \tag{33}$$

Finally, we impose the boundary condition that $u(3, \theta, z) = 100$. This yields

$$100 = \sum_{m=1}^{\infty} c_m I_0(3\pi m/20) \sin(\pi m z/20) \,, \tag{34}$$

which we recognize as a Fourier sine series. Thus, we can solve for the Fourier coefficients, $c_m I_0(3\pi m/20)$,

$$c_m I_0(3\pi m/20) = \frac{1}{10} \int_0^{20} 100 \sin(\pi m z/20) dz = -\frac{200}{m\pi} \cos(\pi m z/20) \bigg|_0^{20}$$

$$= \frac{200}{m\pi} \left[1 - (-1)^m\right] = \begin{cases} 0\,, & \text{for even } m\,, \\ \dfrac{400}{m\pi}\,, & \text{for odd } m\,. \end{cases} \tag{35}$$

This equation determines the c_m, which we can insert back into (33) to obtain the final solution,

$$u(r, \theta, z) = \frac{400}{\pi} \sum_{\text{odd } m} \frac{1}{m I_0(3m\pi/20)} I_0\left(\frac{m\pi r}{20}\right) \sin\left(\frac{m\pi z}{20}\right) \,. \tag{36}$$

4 Boas, p. 644, problem 13.5-12

Solve the Laplace's equation in 2 dimensions in polar coordinates and solve the r and θ equation. Solve the problem of the steady-state temperature in a circular plate if the upper semicircular boundary is held at 100 and the lower at 0.

The Laplacian in polar coordinates is

$$\nabla^2 = \frac{1}{r}\frac{d}{dr}\left(r\frac{d}{dr}\right) + \frac{1}{r^2}\frac{d^2}{d\theta^2} \tag{37}$$

so that Laplace's equation $\nabla^2 u = 0$ becomes (with $u = R(r)\Theta(\theta)$)

$$\frac{r}{R}(rR')' = -\frac{\Theta''}{\Theta} = n^2 \tag{38}$$

The angular dependence is $\Theta = e^{\pm in\theta}$ and n must be an integer for the function to be single-valued, $\Theta(\theta + 2\pi) = \Theta(\theta)$. The radial equation is

$$r^2 R'' + rR' - n^2 R = 0 \tag{39}$$

To solve this we try a solution $R = r^\alpha$ and substituting we find $\alpha = \pm n$. The solution to Laplace's equation is then

$$u = \sum_{n=0} \left[r^n (A_n \sin n\theta + B_n \cos n\theta) + r^{-n}(A'_n \sin n\theta + B'_n \cos n\theta) \right] \tag{40}$$

To solve the requested problem, first we put $A' = B' = 0$ because the temperature would diverge in the center of the plate. Then we apply the boundary condition $u(a, \theta) = 100H(-\theta + \pi)$ where H is the Heaviside step function:

$$u(a, \theta) = 100H(-\theta + \pi) = \sum_{n=0} a^n (A_n \sin n\theta + B_n \cos n\theta) \tag{41}$$

Now we must multiply this equation by $\sin m\theta$ or $\cos m\theta$ and integrate between $(0, 2\pi)$ in order to find respectively A_n and B_n. Because $\int_0^\pi \cos m\theta \, d\theta = \pi \delta_{m0}$, only the constant term in the cosine series is non-zero. It is given by

$$100 \int_0^\pi d\theta = B_0 \int_0^{2\pi} d\theta = 2\pi \implies B_0 = 50 \tag{42}$$

$$a^n A_n = \frac{1}{\pi} \int_0^\pi 100 \sin n\theta \, d\theta = \frac{200}{n\pi} \text{ for odd } n, \text{ and 0 otherwise} \tag{43}$$

We can rewrite the solution as

$$u = 50 + \frac{200}{\pi} \sum_{odd\ n} \left(\frac{r}{a}\right)^n \frac{\sin n\theta}{n} \tag{44}$$

5 Boas, p. 650, problem 13.7-7

Find the steady state temperature distribution inside a sphere of radius 1 with the following surface temperature distribution:

$$u(1, \theta, \phi) = \begin{cases} \cos\theta, & 0 < \theta < \pi/2, \\ 0, & \pi/2 < \theta < \pi. \end{cases} \tag{45}$$

In spherical coordinates, the Laplacian is

$$\nabla^2 = \frac{1}{r^2} \frac{d}{dr} \left(r^2 \frac{d}{dr} \right) + \frac{1}{r^2 \sin\theta} \frac{d}{d\theta} \left(\sin\theta \frac{d}{d\theta} \right) + \frac{1}{r^2 \sin^2\theta} \frac{d^2}{d\phi^2} \tag{46}$$

After separating the variables, $u(r, \theta, \phi) = R(r)\Theta(\theta)\Phi(\phi)$, we have

$$(r^2 R')' = kR\,, \qquad \frac{1}{\sin\theta} \frac{d}{d\theta} \left(\sin\theta \frac{d}{d\theta} \right) \Theta + k\Theta - \frac{m^2}{\sin^2\theta} \Theta = 0\,, \qquad \Phi'' = -m^2 \Phi \tag{47}$$

The R equation is solved by setting $k = l(l + 1)$ and the solutions are r^l, r^{-l-1}. The second solution diverges at the origin so we do not consider it. The Θ equation gives the associated Legendre polynomials in $\cos\theta$, while the Φ equation is solved by the sine and the cosine. The full solution to Laplace's equation is

$$u(r, \theta, \phi) = r^l P_l^m (\cos\theta) e^{\pm im\phi} \tag{48}$$

m must be integer for Φ to be single valued, while l must be integer for P_l^m to not diverge at $\theta = 0, \pi$. As the initial condition (45) has no ϕ-dependence, the only allowed value will be $m = 0$, which gives us the Legendre polynomials $P_l(\cos\theta)$. We can write our solution as

$$u(r, \theta, \phi) = \sum_{l=0}^{\infty} c_l r^l P_l(\cos\theta)\,. \tag{49}$$

We now impose the boundary condition at $r = 1$. Defining $x \equiv \cos\theta$,

$$u(r=1) = \left\{ \begin{array}{ll} x, & 0 < x < q, \\ 0, & -1 < x < 0 \end{array} \right\} = \sum_{l=1}^{\infty} c_l P_l(x) \,. \tag{50}$$

We can now use the orthogonality of the Legendre polynomials, $\int_{-1}^{1} dx\, P_l P_m = \frac{2\delta_{lm}}{2l+1}$ and find

$$c_l = \frac{2l+1}{2} \int_{-1}^{1} u(r=1) P_l(x) dx = \frac{2l+1}{2} \int_0^1 x P_l(x) dx \,. \tag{51}$$

To evaluate this integral, we first employ the recursion relation given in eq (5.8) on p. 570 of Boas,

$$l P_l(x) = (2l-1) x P_{l-1}(x) - (l-1) P_{l-2}(x) \,. \tag{52}$$

Shifting $l \to l+1$, we obtain

$$x P_l(x) = \frac{1}{2l+1} \left[(l+1) P_{l+1}(x) + l P_{l-1}(x) \right] \,. \tag{53}$$

Hence,

$$\int_0^1 x P_l(x) dx = \frac{1}{2l+1} \left[(l+1) \int_0^1 P_{l+1}(x) dx + l \int_0^1 P_{l-1}(x) dx \right] \,. \tag{54}$$

We first consider the cases of $l = 0$ and $l = 1$.

$$\int_0^1 x P_0(x) dx = \int_0^1 x dx = \tfrac{1}{2} \,, \qquad \int_0^1 x P_1(x) dx = \int_0^1 x^2 dx = \tfrac{1}{3} \,. \tag{55}$$

For $l \geq 2$, we make use of the result of problem 12–23.3 on p. 615 of Boas,

$$\int_0^1 P_{2n}(x) dx = 0 \,, \quad \text{for } n > 0 \,, \quad \text{and} \quad \int_0^1 P_{2n+1}(x) dx = \frac{(-1)^n (2n-1)!!}{2^{n+1}(n+1)!} \,. \tag{56}$$

Thus (54) and (56) yield

$$\int_0^1 x P_{2n+1}(x) dx = 0 \,, \qquad \text{for } n = 1, 2, 3, \ldots \,, \tag{57}$$

$$\begin{aligned} \int_0^1 x P_{2n}(x) dx &= \frac{1}{4n+1} \left[\frac{(-1)^n (2n+1)!!}{2^{n+1}(n+1)!} - \frac{(-1)^n (2n-3)!!}{2^{n-1}(n-1)!} \right] \\ &= \frac{(-1)^n (2n-3)!!}{(4n+1) 2^{n+1} (n+1)!} \left[(2n+1)(2n-1) - 4n(n+1) \right] \\ &= \frac{(-1)^{n+1} (2n-3)!!}{2^{n+1}(n+1)!} \,, \qquad \text{for } n = 1, 2, 3, \ldots \end{aligned} \tag{58}$$

Hence, (51) yields

$$c_0 = \tfrac{1}{4}, \qquad c_1 = \tfrac{1}{2} \,, \qquad c_{2n} = \frac{(-1)^{n+1}(4n+1(2n-3)!!}{2^{n+2}(n+1)!} \,, \qquad c_{2n+1} = 0 \,, \qquad \text{for } n = 1, 2, 3, \ldots \tag{59}$$

We can read the first few terms as

$$c_0 = \frac{1}{4}\,, \qquad c_1 = \frac{1}{2}\,, \qquad c_2 = \frac{5}{16}\,, \qquad c_3 = 0\,, \qquad c_4 = -\frac{3}{32}\dots \tag{60}$$

Inserting these results into (49), we obtain

$$u(r,\theta,\phi) = \tfrac{1}{4} + \tfrac{1}{2} r\cos\theta + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}(4n+1(2n-3)!!}{2^{n+2}(n+1)!} r^{2n} P_{2n}(\cos\theta)\,. \tag{61}$$

The first few terms of the series are:

$$u(r,\theta,\phi) = \tfrac{1}{4} + \tfrac{1}{2} r\cos\theta + \tfrac{5}{16} r^2 P_2(\cos\theta) - \tfrac{3}{32} r^4 P_4(\cos\theta) + \cdots\,. \tag{62}$$

6 Boas, p. 650, problem 13.7-11

Find the steady state temperature distribution inside a hemisphere if the spherical surface is held at 100 and the equatorial plane at 0.

We already have the solution for a full sphere with the upper half held at 100 and the lower half at 0 (Boas, p.649, eq. (7.15)):

$$u_0 = \sum_l 100 c_l r^l P_l(\cos\theta), \qquad c_l = \frac{2l+1}{2}\int_{-1}^{1} dx\, f(x) P_l(x)\,, \qquad f(x) = \begin{cases} 0, & -1 < x < 0 \\ 1, & 0 < x < 1 \end{cases} \tag{63}$$

$$c_l = \tfrac{1}{2}\int_{-1}^{1} dx\, f(x)[P'_{l+1} - P'_{l-1}] = \tfrac{1}{2}[P_{l+1} - P_{l-1}]\Big|_0^1 = \tfrac{1}{2}(P_{l-1}(0) - P_{l+1}(0)) = \tag{64}$$

$$= \begin{cases} c_{2l+1} = \frac{1}{2}\frac{(-1)^l(2l-1)!!}{2^l l!} - \frac{1}{2}\frac{(-1)^{l+1}(2l+1)!!}{2^{l+1}(l+1)!} = \frac{1}{2}\frac{(-1)^l(2l-1)!!}{2^l l!}(1 + \frac{2l+1}{2(l+1)}), \\ c_{2l} = 0\,, \qquad c_0 = \frac{1}{2} \end{cases} \tag{65}$$

$$u_0 = 100\big[\tfrac{1}{2}P_0(\cos\theta) + \tfrac{3}{4} r P_1(\cos\theta) - \tfrac{7}{16} r^3 P_3(\cos\theta) + \tfrac{11}{32} r^5 P_5(\cos\theta) + \dots\big] \tag{66}$$

Let u_1 be a solution corresponding to the upper half held at 0 and the lower half at -100:

$$u_1 = \sum_l 100\, d_l r^l P_l(\cos\theta), \qquad d_l = \frac{2l+1}{2}\int_{-1}^{1} dx\, g(x) P_l(x)\,, \qquad g(x) = \begin{cases} -1, & -1 < x < 0 \\ 0, & 0 < x < 1 \end{cases}$$

$$d_l = \frac{2l+1}{2}\int_{-1}^{1} dx\, g(x) P_l(x) = -\frac{2l+1}{2}\int_{-1}^{0} dx\, P_l(x) = -\frac{2l+1}{2}\int_{0}^{1} dx P_l(-x) = -(-1)^l c_l \tag{67}$$

This tells us that $d_0 = -\frac{1}{2},\ d_{2l} = 0,\ d_{2l+1} = c_{2l+1}$: then u_1 is

$$u_1 = 100\big[-\tfrac{1}{2}P_0(\cos\theta) + \tfrac{3}{4} r P_1(\cos\theta) - \tfrac{7}{16} r^3 P_3(\cos\theta) + \tfrac{11}{32} r^5 P_5(\cos\theta) + \dots\big] \tag{68}$$

Finally, $u_0 + u_1$ will still be a solution of Laplace's equation, and satisfy the boundary conditions with the two hemispheres held at 100° and −100°. This also corresponds to have the central plane at 0°, so that the if we consider $u = u_0 + u_1\big|_{\theta<\pi/2}$ we got the solution to our original problem:

$$u = 200\big[\tfrac{3}{4} r P_1(\cos\theta) - \tfrac{7}{16} r^3 P_3(\cos\theta) + \tfrac{11}{32} r^5 P_5(\cos\theta) + \dots\big] \tag{69}$$

7 Boas, p. 651, problem 13.7-16

Separate the wave equation in spherical coordinates and find that the θ, ϕ solutions are the spherical harmonics $Y_l^m(\theta,\phi) = P_l^m(\cos\theta)e^{\pm im\phi}$ and the r solutions are the spherical Bessel functions $j_l(kr)$ and $y_l(kr)$

The wave equation is

$$\nabla^2 u = \frac{1}{v^2}\frac{\partial^2 u}{\partial t^2}\,, \tag{70}$$

where v is the speed of the wave. In spherical coordinates, the Laplacian is

$$\nabla^2 = \frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d}{dr}\right) + \frac{1}{r^2\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d}{d\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{d^2}{d\phi^2} \tag{71}$$

After separating the variables, $u(r,\theta,\phi,t) = F(r,\theta,\phi)T(t) = R(r)\Theta(\theta)\Phi(\phi)T(t)$, we have

$$\frac{\nabla^2 F}{F} = \frac{1}{c^2}\frac{T''}{T} = -k^2 \implies T(t) = e^{\pm ikvt}\,, \qquad \nabla^2 F + k^2 F = 0, \tag{72}$$

$$\frac{1}{R}(r^2R')' + k^2r^2 = -\frac{1}{\Theta\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) + \frac{1}{\Phi\sin^2\theta}\frac{d^2\Phi}{d\phi^2} = l(l+1) \tag{73}$$

which implies that

$$\frac{\sin\theta}{\Theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) = -\frac{1}{\Phi}\frac{d^2\Phi}{d\phi^2} = m^2\,, \qquad (r^2R')' + (k^2r^2 - l(l+1))R\,,$$

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d}{d\theta}\right)\Theta + l(l+1)\Theta - \frac{m^2}{\sin^2\theta}\Theta = 0\,, \qquad \Phi'' = -m^2\Phi \tag{74}$$

The R equation is precisely the spherical Bessel equation in the variable kr, $x^2y''+2xy'+[x^2-l(l+1)]y = 0$ so that its solutions are $j_l(kr)$ and $y_l(kr)$. The Θ equation gives the associated Legendre polynomials in $\cos\theta$, $P_l^m(\cos\theta)$, while the Φ equation is solved by $e^{\pm im\phi}$.

8 Boas, p. 658, problem 13.8-1

Show that the gravitational potential $V = -\frac{Gm}{r}$ satisfy Laplace's equation.

We have to show that $\nabla^2 V \propto \nabla^2\frac{1}{r} = 0$. For that we express the Laplacian in spherical coordinates and note that the only contribution comes from the radial part:

$$\nabla^2\frac{1}{r} = \frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d}{dr}\right)\frac{1}{r} = \frac{1}{r^2}\frac{d}{dr}(-1) = 0\,, \quad \text{for } r \neq 0\,. \tag{75}$$

With this we have proven that the gravitational potential satisfies Laplace's equation at all points in space away from the origin.

9 Boas, p. 658, problem 13.8-3

Solve Poisson's equation for a charge q inside a grounded sphere to obtain the potential V inside the sphere. Sum the series solution and state the image method of solving the problem.

Let the charge q be at $(0,0,a)$ inside the sphere, $a < R$. Poisson's equation is

$$\nabla^2 V = -\pi\rho\,, \qquad \rho = q\delta(x)\delta(y)\delta(z-a) \tag{76}$$

This is solved by

$$u(x,y,z) = \frac{1}{4\pi}\int \frac{\rho(x',y',z')dx'dy'dz'}{\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}} = \frac{q}{\sqrt{x^2+y^2+(z-a)^2}} \tag{77}$$

We still have to satisfy the boundary condition $V(r=R)=0$. For this, we search for another solution satisfying it and add it to our particular solution (77); the solutions to Laplace's equation in spherical coordinates is (as we found earlier solving eq. (47))

$$(A_{lm}r^l + B_{lm}r^{-l-1})P_l^m(\cos\theta)e^{\pm im\phi} \tag{78}$$

As we are searching for a solution in the inside of the sphere, we take $B=0$ so that our result does not diverge at $r=0$. Because the problem is symmetric around the z axis, there will be no ϕ dependence, that is $m=0$. The sum of our two solutions is

$$V = \frac{q}{\sqrt{r^2 - 2ar\cos\theta + a^2}} + \sum_l A_l r^l P_l(\cos\theta) \tag{79}$$

where we have expressed the particular solution (77) in spherical coordinates.

The coefficients are chosen in order to satisfy boundary condition

$$V(r=R) = 0 = \frac{q}{\sqrt{R^2 - 2aR\cos\theta + a^2}} + \sum_l A_l R^l P_l(\cos\theta) = q\sum_l \frac{a^l}{R^{l+1}}P_l(\cos\theta) + \sum_l A_l R^l P_l(\cos\theta),$$

$$\implies A_l = -\frac{qa^l}{R^{2l+1}} \tag{80}$$

where in the first line we have expanded the potential in terms of Legendre polynomials. The solution satisfying the boundary condition is

$$V = \frac{q}{\sqrt{r^2 - 2ar\cos\theta + a^2}} - q\sum_l \frac{a^l r^l}{R^{2l+1}}P_l(\cos\theta) \tag{81}$$

We can sum the second term (using $\sum_l h^l P_l(x) = 1/\sqrt{1-2xh+h^2}$):

$$-q\sum_l \frac{a^l r^l}{R^{2l+1}}P_l(\cos\theta) = -\frac{q}{R}\sum_l \left(\frac{ar}{R^2}\right)^l P_l(\cos\theta) = \frac{-q}{R\sqrt{1-2(ar/R^2)\cos\theta + a^2r^2/R^4}} = \frac{-qR/a}{\sqrt{r^2 - 2(rR^2/a)\cos\theta + R^4/}}$$

This has the same form of (77), but for a charge $\frac{-qR}{a}$ situated at $(0,0,\frac{R^2}{a})$. Then the problem of the grounded sphere with a charge inside it is equivalent to the electrostatic problem of having two charges of different magnitude located at $(0,0,a)$ and $(0,0,\frac{R^2}{a})$. The second charge is called *image charge*.

10 Boas, p. 664, problem 13.10-19

A long conducting cylinder is placed parallel to the z axis in an originally uniform electric field in the negative x direction. The cylinder is held at zero potential. Find the potential in the region outside the cylinder.

First, we recall the solution to Laplace's equation in cylindrical coordinates that we found in Problem 4, eq. (40)

$$V = \sum_{n=0} \left[r^n (A_n \sin n\theta + B_n \cos n\theta) + r^{-n} (A'_n \sin n\theta + B'_n \cos n\theta) \right] = \sum_n r^{-n} (A'_n \sin n\theta + B'_n \cos n\theta) \quad (82)$$

As we will be looking for the potential outside the cylinder, the coefficients multiplying r^n must be zero.

The electric field $\mathbf{E}$ is in the x direction, that is, $\mathbf{E} = -E_0 \mathbf{i} = \nabla V$ so that $V = E_0 x = E_0 r \cos\theta$. We want a solution to Laplace's equation (not Poisson's, as there are no charges) such that $V(r = a) = 0$ and $V = E_0 r \cos\theta$ for large r (far away from the cylinder). Then, we pick

$$V = E_0 r \cos\theta + \sum_n r^{-n} (A_n \cos n\theta + B_n \sin n\theta), \qquad \text{satisfying} \quad V(r = a) = 0 \quad (83)$$

The coefficients are quickly picked by inspection: $B_n = 0$, $A_n = 0$ except for A_1, which satisfies

$$0 = E_0 a \cos\theta + A_1 \frac{1}{a} \cos\theta \implies A_1 = -E_0 a^2 \quad (84)$$

and finally our solution is

$$V = E_0 \left(r - \frac{a^2}{r} \right) \cos\theta \quad (85)$$

11 Boas, p. 664, problem 13.10-20

Use Problem 7 to find the characteristic vibration frequencies of a sound in a spherical cavity.

The vibration modes of a sound in a spherical cavity are solutions to the wave equation (70) subject to the boundary condition $u(r = a) = 0$. The general solution was given by

$$u(r, \theta, \phi, t) = \left\{ \begin{array}{c} j_l(kr) \\ y_l(kr) \end{array} \right\} P_l^m(\cos\theta) e^{\pm im\phi} \left\{ \begin{array}{c} \cos kvt \\ \sin kvt \end{array} \right\} \quad (86)$$

where v is the speed of sound. Because we are looking at the solution inside the sphere, the y_l does not contribute. The boundary condition will be satisfied if

$$j_l(ka) = 0, \text{ that is, if } ka = \lambda_l \text{ is a zero of the spherical Bessel function} \quad (87)$$

The time dependence of the solution is given by sines and cosines, that is, oscillatory modes with an angular frequency $\omega = kv$ so that the frequencies of the normal modes are

$$\nu = \frac{\omega}{2\pi} = \frac{\lambda_l v}{2\pi a} \quad (88)$$

where λ_l is a zero of the spherical Bessel j_l.