1 Boas, p. 734, problem 15.3-9

Two cards are drawn at random from a shuffled deck and laid aside without being examined. Then a third card is drawn. Show that the probability that the third card is spade is $\frac{1}{4}$ just as it was for the first card.

Let us consider all the mutually exclusive possibilities (having 13 spades out of 52 cards):

- if the two discarded cards are spades (probability $= \frac{1}{4} \cdot \frac{12}{51}$: for the first discarded card, there 13 spades out of 52 cards, while for the second we have 12 spades left out of 51 cards), the third could be spades or something else: the probabilities of it being a spade is

$$P_s = \frac{11}{50} \tag{1}$$

as there are 11 spades left in a deck of 50 cards. The probability of this event is then

$$P_2 = \frac{1}{4} \cdot \frac{4}{17} \cdot \frac{11}{50} = \frac{11}{50 \cdot 17}; \tag{2}$$

- if only the first discarded card is a spade (probability $= \frac{1}{4} \cdot (1 - \frac{12}{51})$), we have

$$P_s = \frac{12}{50}, \qquad P_{11} = \frac{1}{4} \cdot \frac{13}{17} \cdot \frac{12}{50} = \frac{13 \cdot 3}{50 \cdot 17}; \tag{3}$$

- if only the second discarded card is a spade (probability $= \frac{3}{4} \cdot \frac{13}{51}$), we have

$$P_s = \frac{12}{50}, \qquad P_{12} = \frac{3}{4} \cdot \frac{13}{51} \cdot \frac{12}{50} = \frac{3 \cdot 13}{50 \cdot 17}; \tag{4}$$

Incidentally, we note that this is the same as in (3), as one expects.

- if none of the two discarded cards are spades (probability $= \frac{3}{4}(1 - \frac{13}{51})$), we have

$$P_s = \frac{13}{50}, \qquad P_0 = \frac{3}{4} \cdot \frac{38}{51} \cdot \frac{13}{50} = \frac{1}{2} \frac{13 \cdot 19}{50 \cdot 17}. \tag{5}$$

The probability for the event given by the sum of these events happening is given by the sum of their probabilities :

$$P = P_2 + P_{11} + P_{12} + P_0 = \frac{1}{4}. \tag{6}$$

as one expects, because taking two cards out of the deck does not add any information about the state of the deck. In fact, the question is the same as asking what is the probability that the third card in the deck is a spade; this is 1/4, as for any other suit, or for any other position inside the deck.

2 Boas, p. 736, problem 15.3-18

Two cards are drawn at random from a shuffled deck:

(a) what is the probability that at least one is a heart?

We can follow the same arguments of Problem 1: having at least one heart corresponds to the first three points in the list, so the probabilities are given by the sum of the probabilities in parentheses before equations (2),(3),(4):

$$P_h = \frac{1}{4} \cdot \frac{12}{51} + \frac{1}{4} \cdot \frac{39}{51} \cdot 2 = \frac{1}{17} + \frac{13}{17 \cdot 2} = \frac{15}{34} \tag{7}$$

(b) If you know that at least one is a heart, what is the probability that both are heart?

Again, we can avoid counting all the states: the probability of having at least one hearth was found in (7), while the probability of having two hearts is $\frac{1}{4} \cdot \frac{12}{51} = \frac{1}{17}$, so that the conditional probability is

$$P_{hh} = \frac{\frac{1}{17}}{\frac{15}{34}} = \frac{2}{15} \tag{8}$$

3 Boas, p. 736, problem 15.3-22

Two players toss a pair of dice trying to get a double (that, is, both dice showing the same number); the first person to toss a double wins. What are the probabilities of winning for the first and for the second player?

The probability of a double in a single toss is

$$P_p = \frac{1}{36} \cdot 6 = \frac{1}{6} \tag{9}$$

One of the player wins if he gets a double before the other one gets it; then

- the first player wins at the first turn with probability $\frac{1}{6}$, at the second turn the first player wins with probability $\frac{5}{6} \cdot (1 - \frac{1}{6}) \cdot \frac{1}{6}$, and so on; the probability is then given by an infinite sum:

$$P_1 = \frac{1}{6} + \frac{5}{6} \cdot \left(1 - \frac{1}{6}\right) \frac{1}{6} + \frac{5}{6} \cdot \left(1 - \frac{1}{6}\right) \frac{5}{6} \cdot \left(1 - \frac{1}{6}\right) \frac{1}{6} + \ldots = \frac{1}{6} \sum_{n=0} \left(\frac{5}{6}\right)^{2n} = \frac{1}{6} \frac{1}{1 - \frac{25}{36}} = \frac{6}{11}, \tag{10}$$

  where we have summed the geometric series, $\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$, for $x < 1$.

- the second player wins at the first turn if the first player has not one yet and he gets a double (probability $= \frac{5}{6} \cdot \frac{1}{6}$) and so on: the probability of him winning is

$$P_2 = \frac{5}{6} \cdot \frac{1}{6} + \frac{5}{6} \cdot \left(1 - \frac{1}{6}\right) \frac{5}{6} \cdot \frac{1}{6} + \ldots = \frac{5}{6} P_1 = \frac{5}{11}, \tag{11}$$

  as expected. Indeed we could have immediately concluded that $P_2 = 1 - P_1$ without an explicit computation.

In this kind of game, the second player always starts with an handicap; in the case of tossing a coin (two possible outcomes), the probabilities would have been $\frac{2}{3}$, $\frac{1}{3}$. We see that by having a larger number of possible outcomes, the advantage of the first player reduces; in general, for a throw of an object with n outcomes, we would have

$$P_1 = \frac{1}{n} \sum_{k=0}^{\infty} \left(1 - \frac{1}{n}\right)^{2k} = \frac{n}{2n-1} , \qquad P_2 = \frac{n-1}{2n-1} \tag{12}$$

so that the game is fair for large n, specifically $P_1 = P_2$ for $n \to \infty$.

4 Boas, p. 743, problem 15.4-9

Two cards are drawn at random from a shuffled deck. What is the probability that both are red? If at least one is red, what is the probability that both are red? If at least one is a red ace, what is the probability that both are red? If exactly one is a red ace, what is the probability that both are red?

The probability of both cards being red is

$$P_{rr} = \frac{26}{52} \cdot \frac{25}{51} = \frac{25}{102} \tag{13}$$

because there are 26 red cards out 52 when we pick the first card, and 25 out of 51 for the second.

For the following questions, we will make use of the conditional probability : the probability of the event B happening, provided that A happened, is[1]

$$P(B|A) = \frac{P(B \cap A)}{P(A)} \tag{14}$$

If at least one is red, the probability that both are red is

$$P_{rr}^{r} = \frac{\frac{26}{52} \cdot \frac{25}{51}}{\frac{26}{52}(1 - \frac{25}{51}) \cdot 2 + \frac{26}{52} \cdot \frac{25}{51}} = \frac{25}{77} , \tag{15}$$

where in the denominator we have the probability of picking at least a red card (i.e., either one or two red cards), and in the numerator we have the probability of having two red cards.

If at least one is a red ace, the probability that both are red is

$$P_{rr}^{a} = \frac{\frac{2}{52} \cdot \frac{25}{51} + \frac{24}{52} \cdot \frac{2}{51}}{\frac{2}{52}(1 - \frac{1}{51}) \cdot 2 + \frac{2}{52} \cdot \frac{1}{51}} = \frac{98}{2 \cdot (100+1)} \frac{49}{101} \tag{16}$$

where in the denominator we have the probability of picking at least a red ace (2 cards out of 52), that is, one or two red aces, and in the numerator we have the probability of having a red ace and a red card.

If exactly one is a red ace, the probability that both are red is

$$P_{rr}^{a} = \frac{\frac{2}{52} \cdot \frac{24}{51} + \frac{24}{52} \cdot \frac{2}{51}}{\frac{2}{52}(1 - \frac{1}{51}) \cdot 2} = \frac{96}{2 \cdot 100} = \frac{12}{25} \tag{17}$$

where in the denominator we have the probability of picking one red ace (2 cards out of 52), and in the numerator we have the probability of having a red ace and a red card which is not a red ace.

[1] In the notation of Boas, $P_A(B) = P(AB)/P(A)$.

5 Boas, p. 743, problem 15.4-10

What is the probability that you and a friend have different birthdays? (For simplicity, let a year have 365 days) What is the probability that three people have different birthdays? Show that the probability that n people have n different birthdays is

$$p = \left(1 - \frac{1}{365}\right)\left(1 - \frac{2}{365}\right)\left(1 - \frac{3}{365}\right)\dots\left(1 - \frac{n-1}{365}\right) \tag{18}$$

Estimate this for $n \ll 365$ by calculating $\ln p$. Find the smallest n for which $p < \frac{1}{2}$. Hence show that for a group of 23 people or more, the probability is greater than $\frac{1}{2}$ that two of them have the same birthday.

The probability that two people have different birthdays is given by

$$p_2 = \left(1 - \frac{1}{365}\right) \tag{19}$$

as this is the opposite event of having the same birthday, which has probability 1/365. If there are three people, the probability is given by the probability that two of them have different birthdays, multiplied by the probability that the third has another birthday:

$$p_3 = \left(1 - \frac{1}{365}\right)\left(1 - \frac{2}{365}\right) \tag{20}$$

where we have 2/365 as there are 2 days out of 365 which already are birthdays. Given n people, we can prove that the formula (18) is right: this can be proven by induction. The cases $n = 2, 3$ have already been proved, so let us see that if the formula holds for $n-1$ people then it will hold also for n.

The probability that n people have different birthdays is given by the probability that $n-1$ people have different birthdays times the probability that the n-th one has another different birthday:

$$p_n = p_{n-1} \cdot \left(1 - \frac{n-1}{365}\right) = \left(1 - \frac{1}{365}\right)\left(1 - \frac{2}{365}\right)\left(1 - \frac{3}{365}\right)\dots\left(1 - \frac{n-1}{365}\right), \tag{21}$$

where in the first passage we have $\frac{n-1}{365}$ because $n-1$ days out of 365 are already occupied, and in the last step we have used the inductive hypothesis.

We can expand this expression in terms of $\frac{n}{365}$, for small n. First, let us compute the logarithm of p, and use the property $\ln abc\dots = \ln a + \ln b + \ln c + \dots$, as well as the expansion $\ln(1+x) = x + \mathcal{O}(x^2)$:

$$\ln p_n \simeq -\frac{1}{365} - \frac{2}{365} - \frac{3}{365} - \dots - \frac{n-1}{365} = -\frac{1}{365}\sum_{k=1}^{n-1} k = -\frac{1}{365}\frac{n(n-1)}{2} \tag{22}$$

where in the last step we have summed the arithmetic series. We note that $\ln p_n < 0$ as expected since $p_n < 1$. We can write the probability as

$$p_n \approx \exp\left[-\frac{1}{365}\frac{n(n-1)}{2}\right] \tag{23}$$

We can solve to see when $p_n > \frac{1}{2}$:

$$\frac{1}{365}\frac{n(n-1)}{2} > \ln 2, \quad \Longrightarrow \quad n^2 - n - 730 \ln 2 > 0\,, \tag{24}$$

which implies that

$$n < n_1 \text{ or } n > n_2, \qquad \text{with} \quad n_{1,2} = \frac{1 \pm \sqrt{1 + 4 \cdot 730 \ln 2}}{2} = \begin{cases} n_1 = -21.99994 \\ n_2 = 22.99994 \end{cases} \tag{25}$$

Since only positive integer values of n are relevant, we conclude that $p_n > \frac{1}{2}$ for $n \geq 23$. This tells us that in a group of 23 people there a 50% chance to have two people sharing the same birthday. Note that our approximation of $\ln(1+x) = x + \mathcal{O}(x^2) \simeq x$ is reliable, since the correction to this result is of order $(\frac{n}{365})^2 < 1\%$.

6 Boas, p. 743, problem 15.4-13

Generalize Example 3 to show that the number of ways of putting N balls in n boxes with N_1 in box 1, N_2 in box 2, etc., is

$$\left(\frac{N!}{N_1! N_2! \ldots N_n!} \right) \tag{26}$$

Following Example 3, p. 739, we want to put N balls in n boxes: let N_i be the number of balls that we want to put in the i-th box:

$$\text{Number of balls:} \quad N_1 \quad N_2 \quad N_3 \quad \ldots \quad N_n \tag{27}$$

$$\text{in box number:} \quad 1 \quad 2 \quad 3 \quad \ldots \quad n\ . \tag{28}$$

First we ask how many ways we can select N_1 balls (out of N) to go in the first box: this is $C(N, N_1)$, the number of combinations of N balls taken N_1 at a time (where we do not consider the order of the balls). Then we have $N - N_1$ balls left, which gives us $C(N - N_1, N_2)$ different combinations. Going on this way, we have that the number of putting N balls in n boxes is

$$\begin{aligned} & C(N, N_1) \cdot C(N - N_1, N_2) \cdot C(N - N_1 - N_2, N_3) \cdot \ldots \cdot C(N_n, N_n) = \\ & = \frac{N!}{(N - N_1)! N_1!} \cdot \frac{(N - N_1)!}{(N - N_1 - N_2)! N_2!} \cdot \frac{(N - N_1 - N_2)!}{(N - N_1 - N_2 - N_3)! N_3!} \cdot \ldots \cdot \frac{N_n!}{N_n! 0!} = \frac{N!}{N_1! N_2! \ldots N_n!} \end{aligned} \tag{29}$$

7 Boas, p. 744, problem 15.4-22

Suppose 13 people want to schedule a regular meeting one evening a week. What is the probability that there is an evening when everyone is free if each person is already busy one evening a week?

Given that each person is busy one day per week, the probability that the person is not busy on a given day is 6/7. As there are 13 people, the probability for all thirteen people to be not busy on the same day is

$$\left(\frac{6}{7} \right)^{13} = 0.1348\,. \tag{30}$$

In fact, this is the incorrect answer given the wording of the problem. I believe that Boas meant to ask for the probability that there is a *specified* evening when everyone is free, say on Mondays. In this case, the answer given in (30) would be correct.

By not specifying a particular evening, (30) does not take into account that the day that all people are simultaneously free can be any of the seven days. However, it is incorrect to simply multiply the probability above by 7, since you would be double counting by not taking properly into account the possibility that there may be more than one day in which all people are not busy. The easiest way to convince yourself of this is to consider an example where the numbers are smaller. Take the case of four people and three days. You can enumerate all the possibilities fairly easily. If you did not distinguish among people then there are 15 ways to identify one of three days in which each of four people are busy (as this is analogous to finding the arrangements of 4 indistinguishable balls in three distinguishable boxes). But, in this problem, the people are distinguishable. So, for each of the 15 cases you must determine a multiplicity. It is straightforward to verify by brute force that when you take into account the multiplicity, there are $3^4 = 81$ total possibilities. Of these, there are 16 cases in which no one is busy on day one, and similarly for day 2 and day 3. Note that $(2/3)^4 = 16/81$ which would be the correct probability for no one to be busy on a *particular* day, say day 1. To find the probability that no one is busy on *either* day 1, day 2 or day 3, one should multiply by 3 but then correct for the fact that in three cases there are two days in which no one is busy. Hence, it follows that in this simple example, there are 45 cases in which no one is busy on at least one of the three days, with probability $45/81 = 5/9$.

The solution to Boas' problem as stated requires a significant amount of analysis due to the larger numbers involved. Here is one method for solving this problem. It is convenient to think of this problem as a balls/boxes problem—if each person is a ball and the day that he or she is busy is represented by one of seven boxes, then the number of possible schedules is equal to the number of ways distributing 13 distinguishable balls into 7 distinguishable boxes. In some of these arrangements, one or more of the boxes will be empty. The problem asks for the probability P that there is an evening when everyone is free, which is equal to the ratio of the number of arrangements where there is at least one box empty divided by the total number of arrangements. In fact, it is easier to compute $1 - P$ which is equal to the number of arrangements in which no box is empty divided by the total number of arrangements.

In class, we showed that the number of ways of distributing r *indistinguishable* balls into N distinguishable boxes is equal to $C(N+r-1, r)$. But, in this problem, the balls should be considered distinguishable in which case there are N^r possible arrangements, as in Example 3 presented on pp. 739–740 of Boas. We wish to compute the number of arrangements in which at least one box is empty. If this number is $N_{\rm empty}$, then the corresponding probability is $P = N_{\rm empty}/N^r$. It is somewhat easier to count the number of arrangements in which none of the boxes are empty. This number would be $N_{\rm non-empty} = N^r - N_{\rm empty}$, so that the probability P can be written as:

$$P = 1 - \frac{N_{\rm non-empty}}{N^r}\,. \tag{31}$$

In the case of $N = 7$ and $r = 13$, we can enumerate explicitly all the possible arrangements in which none of the boxes are empty:

arrangement	multiplicative factor
$7,1,1,1,1,1,1$	$\times\, 7$
$6,2,1,1,1,1,1$	$\times\, 42$
$5,3,1,1,1,1,1$	$\times\, 42$
$5,2,2,1,1,1,1$	$\times\, 105$
$4,4,1,1,1,1,1$	$\times\, 21$
$4,3,2,1,1,1,1$	$\times\, 210$
$4,2,2,2,1,1,1$	$\times\, 140$
$3,3,3,1,1,1,1$	$\times\, 35$
$3,3,2,2,1,1,1$	$\times\, 210$
$3,2,2,2,2,1,1$	$\times\, 105$
$2,2,2,2,2,2,1$	$\times\, 7$

The multiplicative factor takes into account the additional arrangements not explicitly written out. For example, the first line above corresponds to the seven orderings $(7,1,1,1,1,1,1)$, $(1,7,1,1,1,1,1)$, $(1,1,7,1,1,1,1)$, $(1,1,1,7,1,1,1)$, $(1,1,1,1,7,1,1)$, $(1,1,1,1,1,7,1)$ and $(1,1,1,1,1,1,7)$. As an example, given $(4,3,2,1,1,1,1)$, we can place the 4 in seven possible positions, the 3 in six possible positions and the 2 in five possible positions, which yields $7\cdot 6\cdot 5 = 210$ possible orderings (since the location of the 1's are fixed once these choices have been made). For $(5,2,2,1,1,1,1)$, we must account for the fact that the two 2's are indistinguishable; hence the number of orderings is $7\cdot 6\cdot 5/2! = 105$. For the case of $(3,3,2,2,1,1,1)$ we would get $7\cdot 6\cdot 5\cdot 4/(2!\cdot 2!) = 210$ taking into account the indistinguishability of the two pairs of 2's, etc.

Using the result of problem 6 of this problem set, the number of ways of putting r balls in N boxes with r_1 balls in box 1, r_2 balls in box 2, etc. is given by

$$\text{Multinomial}[r_1, r_2, \ldots, r_n] = \frac{r!}{r_1! r_2! \cdots r_N!}\,, \tag{32}$$

where $r_1 + r_2 + \cdots + r_N = r$. Employing the table above, we conclude that

$$\begin{aligned} N_{\text{non-empty}} &= 7\,\text{Multinomial}[7,1,1,1,1,1,1] + 42\,\text{Multinomial}[6,2,1,1,1,1,1] \\ &\quad +42\,\text{Multinomial}[5,3,1,1,1,1,1] + 105\,\text{Multinomial}[5,2,2,1,1,1,1] \\ &\quad +21\,\text{Multinomial}[4,4,1,1,1,1,1] + 210\,\text{Multinomial}[4,3,2,1,1,1,1] \\ &\quad +140\,\text{Multinomial}[4,2,2,2,1,1,1] + 35\,\text{Multinomial}[3,3,3,1,1,1,1] \\ &\quad +210\,\text{Multinomial}[3,3,2,2,1,1,1] + 105\,\text{Multinomial}[3,2,2,2,2,1,1] \\ &\quad +7\,\text{Multinomial}[2,2,2,2,2,2,1]\,. \\ &= 28805736960\,, \end{aligned}$$

with a little help from Mathematica. Using

$$7^{13} = 96889010407\,, \tag{33}$$

it follows that

$$P = 1 - \frac{28805736960}{96889010407} = 1 - 0.297307 = 0.702693\,. \tag{34}$$

That is, there is a 70% chance that there is an evening free if each person is busy one evening a week. This is quite different from (30), and is due to the fact that we are accounting for the possibility that the free night can be any of the seven days of the week.

This was a tedious computation! In a more advanced course on combinatorics, you would learn that the number of number of ways to distribute k distinguishable objects into N distinguishable boxes under the assumption that no box is left empty is given by:

$$N!\,S(r,N) \equiv \sum_{k=0}^{N} (-1)^k \binom{N}{k} (N-k)^r\,, \tag{35}$$

where $S(r,N)$ is called the Stirling number of the second kind. Using this notation, the answer to Boas' problem as originally worded is

$$P = 1 - \frac{7!\,S(13,7)}{7^{13}}\,. \tag{36}$$

However, I am quite certain that it was not the intention of Boas to lead you in this direction!

8 Boas, p. 749, problem 15.5-7

A weighted coin with probability p of coming down heads is tossed three times. Let the random variable be x = number of heads minus number of tails. Set up the sample space, make a table of the different values of x and the associated probabilities; compute the mean, variance and standard deviation of x, and the cumulative distribution function.

All the outcomes for this three tosses of a coin are

Outcome	HHH	HHT	HTH	HTT	THH	THT	TTH	TTT
Prob.	p^3	$p^2(1-p)$	$p^2(1-p)$	$p(1-p)^2$	$p^2(1-p)$	$p(1-p)^2$	$p(1-p)^2$	$(1-p)^3$

(37)

The sample space for x is given by

$x = N_H - N_T$	3	1	-1	-3
Prob.	p^3	$3p^2(1-p)$	$3p(1-p)^2$	$(1-p)^3$

(38)

The mean, variance and standard deviation of x are

$$E(x) = \sum_i x_i p_i = 3[p^3 + p^2(1-p) - p(1-p)^2 - (1-p)^3] = 3(2p-1) \tag{39}$$

$$\mathrm{Var}(x) = \sum_i (x_i^2 - E(x)^2) p_i = [9p^3 + 3p^2(1-p) + 3p(1-p)^2 + 9(1-p)^3] - 9(2p-1)^2 = 12p(1-p)$$

$$\sigma_x = \sqrt{\mathrm{Var}(x)} = 2\sqrt{3p(1-p)} \tag{40}$$

We can check that our results are correct by seeing that $\sigma^2 = E(x^2) - [E(x)]^2$:

$$\sigma^2 = 12p(1-p) = [9p^3 + 3p^2(1-p) + 3p(1-p)^2 + 9(1-p)^3] - 9(2p-1)^2 = E(x^2) - E(x)^2 \tag{41}$$

The cumulative distribution function is

x	3	1	-1	-3
$F(x)$	p^3	$3p^2 - 2p^3$	$p^3 - 3p^2 + 3p$	1

(42)

9 Boas, p. 749, problem 15.5-8

Would you pay \$10 per throw of two dice if you were to receive a number of dollars equal to the product of the numbers on the dice?

The random variable we are interested in is the product of the two numbers on the dice; if its expectation value is more than 10, the game is favorable. The sample space is the one created by the toss of two dice:

Product	1	2	3	4	5	6	8	9	10	12	15	16	18	20	24	25	30	36
Prob.	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{4}{36}$	$\frac{2}{36}$	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{4}{36}$	$\frac{2}{36}$	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{2}{36}$	$\frac{2}{36}$	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{1}{36}$

(43)

Then, the expectation value is

$$\bar{x} = \sum_i x_i p_i = \frac{1}{36}(1 + 4 + 6 + 12 + 10 + 24 + 16 + 9 + 20 + 48 + 30 + 16 + 36 + 40 + 48 + 25 + 60 + 36) =$$

$$= \frac{441}{36} = 12.25 \tag{44}$$

As the expectation value is bigger than 10, the game is favorable. In average, you will earn \$2.25 at each game.

10 Boas, p. 749, problem 15.5-9

Show that the expectation of the sum of two random variables defined over the same sample space is the sum of the expectations.

In a sample space of n points, the random variables x and y take on the values $x_1, x_2, \ldots, x_n$ and $y_1, y_2, \ldots, y_n$, respectively. The corresponding probability distributions for x and y are denoted by p_j and p_k, respectively, where the indices j and k can take on any integer value between 1 and n. The expectation values for the variables x and y are

$$E(x) = \sum_j x_j p_j \,, \qquad\qquad E(y) = \sum_k y_k p_k \,. \tag{45}$$

Let p_{jk} be the joint probability distribution. Then,

$$E(x+y) = \sum_j \sum_k (x_j + y_k) p_{jk} \,. \tag{46}$$

Recall that the marginal probability distributions are given by:

$$p_j = \sum_k p_{jk} \quad \text{and} \quad p_k = \sum_j p_{jk} \,. \tag{47}$$

Hence,

$$E(x+y) = \sum_j \sum_k (x_j + y_k) p_{jk} = \sum_j x_j \sum_k p_{jk} + \sum_k y_k \sum_j p_{jk} = \sum_j x_j p_j + \sum_k y_k p_k = E(x) + E(y) \,. \tag{48}$$

11 Boas, p. 754, problem 15.6-2

It is shown in the kinetic theory of gases that the probability for the distance a molecule travels between collisions to be between x and $x + dx$, is proportional to $e^{-x/\lambda} dx$, where λ is a constant. Show that the average distance between collisions is λ. Find the probability for a free path of length $\geq 2\lambda$.

First, we are told that the probability density is proportional to $e^{-x/\lambda}$. Let us set the proportionality constant by checking that the probability for the nucleus to travel any distance is 1:

$$1 = \int_0^\infty p(x) dx = \int_0^\infty C e^{-x/\lambda} dx = C\lambda \qquad \Longrightarrow \qquad p(x) = \frac{1}{\lambda} e^{-x/\lambda} \tag{49}$$

The average distance without collisions is then

$$\overline{x} = \int x p(x) dx = \int_0^\infty \frac{x}{\lambda} e^{-x/\lambda} dx = -x e^{-x/\lambda} \Big|_0^\infty + \int_0^\infty e^{-x/\lambda} dx = \lambda \,; \tag{50}$$

the parameter λ in the probability density is precisely the mean path the nucleus travels without being hit (the mean free path). The probability for having a free path of length $\geq 2\lambda$ is given by

$$P(x \geq 2\lambda) = \int_{2\lambda}^\infty p(x) dx = \int_{2\lambda}^\infty \frac{e^{-x\lambda}}{\lambda} dx = e^{-2} \approx 0.135 \tag{51}$$

12 Boas, p. 755, problem 15.6-6

A circular garden bed of radius 1 m is to be planted so that N seeds are uniformly distributed over the circular area. Then we can talk about the number n of seeds in some particular area A, or we call n/N the probability for any one particular seed to be in the area A. Find the probability $F(r)$ that a seed is within r of the center. Find $f(r)dr$, the probability for a seed to be between r and $r+dr$ from the center. Find $\bar{r}$ and σ.

If the seeds are uniformly distributed, the number of seed will grow with the area:

$$n(r) = C \cdot \pi r^2 , \qquad n(1) = N = C\pi \implies n(r) = Nr^2 \tag{52}$$

The probability for any particular seed to be within a distance r from the center is

$$F(r) = \frac{n}{N} = r^2 . \tag{53}$$

The probability density is then

$$f(r) = \frac{d}{dr}F(r) = 2r \tag{54}$$

which gives the probability for a seed to be between r and $r+dr$, $f(r)dr$. The mean and standard deviation are:

$$\overline{r} = \int_0^1 rf(r)dr = \int_0^1 2r^2 dr = \frac{2}{3} \tag{55}$$

$$\sigma = \sqrt{\langle (r-\overline{r})^2 \rangle} = \sqrt{\langle r^2 \rangle - \overline{r}^{\,2}}, \qquad \langle r^2 \rangle = \int_0^1 r^2 f(r)dr = \frac{1}{2}; \tag{56}$$

$$= \frac{1}{\sqrt{18}} = \frac{\sqrt{2}}{6}. \tag{57}$$