Problem 1

Solve the differential equation:

$$xy'' + (\tfrac{1}{2} - x)y' - \tfrac{1}{2}y = 0, \tag{1}$$

by expanding the solution in a generalized power series:

(a) Determine the generalized power series for the two linearly independent solutions of eq. (1). Show that one of the two series solutions, denoted by $y_1(x)$, is an elementary function by explicitly summing the series.

We insert the power series $y(x) = \sum_{n=0}^{\infty} a_n x^{n+s}$:

$$\sum_{n=0}^{\infty}(n+s)(n+s-1)a_n x^{n+s-1} + \tfrac{1}{2}\sum_{n=0}^{\infty}(n+s)a_n x^{n+s-1} - \sum_{n=0}^{\infty} a_n(n+s)x^{n+s} - \tfrac{1}{2}\sum_{n=0}^{\infty} a_n x^{n+s} = 0\,. \tag{2}$$

We can rewrite (2) as:

$$\begin{aligned}\left[s(s-1) + \tfrac{1}{2}s\right] a_0 x^{s-1} + \sum_{n=1}^{\infty}(n+s)(n+s-1)a_n x^{n+s-1} + \tfrac{1}{2}\sum_{n=1}^{\infty}(n+s)a_n x^{n+s-1} \\ - \sum_{n=0}^{\infty} a_n(n+s)x^{n+s} - \tfrac{1}{2}\sum_{n=0}^{\infty} a_n x^{n+s} = 0\,.\end{aligned} \tag{3}$$

The sums can be combined by defining $m = n-1$ in the first two summations and $m = n$ in the last two summations. In this case, all the sums will run from $m = 0$ to $m = \infty$. Hence, (2) can be rewritten as:

$$\begin{aligned}&\left[s(s-1) + \tfrac{1}{2}s\right] a_0 x^{s-1} \\ &\quad + \sum_{m=0}^{\infty}\left[(m+s+1)(m+s)a_{m+1} + \tfrac{1}{2}(m+s+1)a_{m+1} - (m+s)a_m - \tfrac{1}{2}a_m\right] x^{m+s} = 0\,.\end{aligned} \tag{4}$$

The indicial equation is obtained from the coefficient of the smallest power of x—in this case the coefficient of x^{s-1}. Setting this coefficient to zero yields

$$s(s-1) + \tfrac{1}{2}s = 0 \implies s = 0\,, \tfrac{1}{2}\,. \tag{5}$$

Setting the coefficient of x^{m+s} to zero for $m = 0, 1, 2, 3, \ldots$ yields the recursion relation for the coefficients a_m.

$$(m+s+1)(m+s)a_{m+1} + \tfrac{1}{2}(m+s+1)a_{m+1} - (m+s)a_m - \tfrac{1}{2}a_m = 0\,, \tag{6}$$

which yields

$$a_{m+1} = \frac{a_m}{m+s+1}\,, \quad \text{for } m = 0, 1, 2, 3, \ldots \tag{7}$$

We now consider the two possible indicial roots, $s = 0$ and $s = \tfrac{1}{2}$.

- $s = 0$: the recursion relation becomes

$$a_{m+1} = \frac{a_m}{m+1} \tag{8}$$

which implies that

$$a_{m+1} = \frac{1}{(m+1)m} a_{m-1} = \frac{1}{(m+1)m(m-1)} a_{m-2} = \cdots = \frac{1}{(m+1)!} a_0 \,. \tag{9}$$

The first solution then is

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^n = a_0 \sum_{n=0}^{\infty} \frac{x^n}{n!} = a_0 e^x \,. \tag{10}$$

- $s = \frac{1}{2}$: the recursion relation becomes

$$a_{m+1} = \frac{a_m}{m + \frac{3}{2}} \,, \tag{11}$$

which implies that

$$a_{m+1} = \frac{1}{m+\frac{3}{2}} a_n = \frac{2}{2m+3} a_m = \frac{4}{(2m+3)(2m+1)} a_{m-1} = \cdots = \frac{2^{m+1}}{(2m+3)!!} a_0 \,. \tag{12}$$

The second solution is then

$$y_2(x) = \sum_n a_n x^{n+\frac{1}{2}} = a_0 x^{\frac{1}{2}} \sum_n \frac{(2x)^n}{(2n+1)!!} \tag{13}$$

(b) Inserting $y = y_1(x)v(x)$ with $v(x) = \int dx\, w(x)$, into eq. (1) [and using the fact that $y_1(x)$ satisfies eq. (1)], find the equation that $w(x)$ satisfies, and solve it.

$$y_2 = y_1(x)v(x) : \tag{14}$$
$$x(y_1''v + 2y_1'v' + y_1 v'') + (\tfrac{1}{2} - x)(y_1'v + y_1 v') - \tfrac{1}{2} y_1 v = 0, \tag{15}$$
$$x y_1 v'' + (2xy_1' + (\tfrac{1}{2} - x)y_1)v' = 0 \implies x y_1 w' + (2xy_1' + (\tfrac{1}{2} - x)y_1)w = 0; \tag{16}$$

The known solution is $y_1 = e^x$, so that $y_1' = y_1$ and the equation for w becomes

$$xw' + (\tfrac{1}{2} + x)w = 0 \implies \frac{w'}{w} = -1 - \frac{1}{2x} \implies w(x) = c\frac{1}{\sqrt{x}} e^{-x} \tag{17}$$

(c) EXTRA CREDIT: Evaluate the indefinite integral of $w(x)$ in terms of one of the special functions that we studied in Physics 116A.

The integral is

$$v(x) = c \int x^{-1/2} e^{-x} dx = 2c \int e^{-z^2} dz \,, \tag{18}$$

after a change of variables $x = z^2$ (so that $dx = 2zdz = 2x^{1/2}dz$ or $dz = \frac{1}{2}x^{-1/2}dx$). Recall the definition of the error function,

$$\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt \,. \tag{19}$$

Hence, $v(x) = c\sqrt{\pi}\,\mathrm{erf}(\sqrt{x}) + c'$, where c' is a constant of integration. Finally, $y_2(x) = y_1(x)v(x)$, where $y_1(x) = e^x$ was obtained above (the overall constant can be absorbed into the definition of c). Clearly, we can ignore the constant of integration c', as it just gives a multiple of the first solution. Thus, the second solution is:

$$y_2(x) = \tfrac{1}{2}\sqrt{\pi}c_0 e^x \mathrm{erf}(\sqrt{x})\,, \tag{20}$$

where we have set $c = \frac{1}{2}c_0$ such that the above solution matches precisely with the series solution given in (13). (You can easily verify this claim by plugging in the series solutions for e^x and $\mathrm{erf}(\sqrt{x})$ and multiplying them together.)

Problem 2

The spherical modified Bessel functions are defined by:

$$i_n(x) = \sqrt{\frac{\pi}{2x}} I_{n+\frac{1}{2}}(x), \tag{21}$$

$$k_n(x) = \sqrt{\frac{2}{\pi x}} K_{n+\frac{1}{2}}(x), \tag{22}$$

where n is an integer. $I_{n+\frac{1}{2}}(x)$ and $K_{n+\frac{1}{2}}(x)$ are modified Bessel functions of order $n + \frac{1}{2}$.

(a) Express $i_0(x)$ and $k_0(x)$ explicitly in terms of elementary functions. Cite any pertinent formulae that you take from Boas.

From Boas, p 595, we take the definitions of the modified Bessel functions:

$$I_p(x) = i^{-p} J_p(ix)\,, \qquad K_p(x) = \frac{\pi}{2} i^{p+1} H_p^{(1)}(ix), \qquad H_p^{(1)}(x) = J_p(x) + iY_p(x). \tag{23}$$

Also from Boas p. 596 we take the expressions of the spherical Bessel functions in terms of elementary functions

$$j_n(x) = \sqrt{\frac{\pi}{2x}} J_{n+\frac{1}{2}}(x) = x^n \left(-\frac{1}{x}\frac{d}{dx}\right)^n \left(\frac{\sin x}{x}\right), \tag{24}$$

$$y_n(x) = \sqrt{\frac{\pi}{2x}} Y_{n+\frac{1}{2}}(x) = -x^n \left(-\frac{1}{x}\frac{d}{dx}\right)^n \left(\frac{\cos x}{x}\right); \tag{25}$$

To find i_0, k_0 we simply substitute the expressions:

$$i_0(x) = \sqrt{\frac{\pi}{2x}} I_{\frac{1}{2}}(x) = \sqrt{\frac{\pi}{2x}} i^{-\frac{1}{2}} J_{\frac{1}{2}}(ix) = \sqrt{\frac{\pi}{2ix}} J_{\frac{1}{2}}(ix) = j_0(ix) = \frac{\sin ix}{ix} = \frac{\sinh x}{x} \tag{26}$$

$$\begin{aligned} k_0(x) &= \sqrt{\frac{2}{\pi x}} K_{\frac{1}{2}} = \sqrt{\frac{2}{\pi x}} \frac{\pi}{2} i^{\frac{3}{2}} H^{(1)}_{\frac{1}{2}}(ix) = -\sqrt{\frac{\pi}{2ix}} H^{(1)}_{\frac{1}{2}}(ix) \\ &= -\sqrt{\frac{\pi}{2ix}} \left(J_{\frac{1}{2}}(ix) + iY_{\frac{1}{2}}(ix)\right) = -(j_0(ix) + iy_0(ix)) = -\left(\frac{\sin ix}{ix} - i\frac{\cos ix}{ix}\right) \\ &= \frac{\cos ix + i \sin ix}{x} = \frac{e^{i(ix)}}{x} = \frac{e^{-x}}{x}\,. \end{aligned} \tag{27}$$

(b) Compute the Wronskian of $k_0(x)$ and $i_0(x)$. Are these two functions linearly independent?

The Wronskian is

$$W(i_0, k_0) = \det\begin{pmatrix} i_0 & k_0 \\ i_0' & k_0' \end{pmatrix} = i_0 k_0' - k_0 i_0' = -\frac{\sinh x}{x}\left(\frac{e^{-x}}{x} + \frac{e^{-x}}{x^2}\right) - \frac{e^{-x}}{x}\left(\frac{\cosh x}{x} - \frac{\sinh x}{x^2}\right)$$

$$= -\frac{e^{-x}}{x^2}\Big[\sinh x + \cosh x\Big] + \frac{e^{-x}}{x^3}\Big[-\sinh x + \sinh x\Big] = -\frac{1}{x^2}\,. \tag{28}$$

The Wronskian is non-zero and the functions are linearly independent.

Problem 3

In quantum mechanics, the Schrodinger equation for a free particle in one dimension is

$$\frac{\partial^2 \psi(x,t)}{\partial x^2} = -i\frac{\partial \psi(x,t)}{\partial t}, \tag{29}$$

where the (complex) wave function $\psi(x,t)$ provides information on the probability that the particle is located in the vicinity of x at time t.

Using the separation of variables technique, solve eq. (29) subject to the boundary conditions $\psi(0,t) = \psi(1,t) = 0$ and the initial condition $\psi(x,0) = 1$ $(0 < x < 1)$. These conditions correspond to the physical situation of a particle trapped in a "box" (more precisely, an interval) of length 1, whose location at time $t = 0$ is equally probable anywhere in the interval $0 < x < 1$.

HINT: Your final solution for $\psi(x,t)$ should take the form of an infinite sum over solutions to eq. (29) that satisfy the boundary conditions. Make sure that you impose the initial condition to determine all remaining unknown coefficients.

For our solution we assume the form $\psi(x,t) = X(x)T(t)$. Schrödinger equation becomes

$$X''(x)T(t) = -iX(x)T'(t)\,, \qquad \frac{X''}{X}(x) = -i\frac{T'}{T}(t) = -k^2\,; \tag{30}$$

$$T(t) = e^{-ik^2 t}\,, \qquad X(x) = A\cos kx + B\sin kx. \tag{31}$$

From the boundary condition $\psi(0,t) = 0$ we have $A = 0$, while $\psi(1,t) = 0$ implies $\sin k = 0$, that is, $k = k_n = n\pi$. Then

$$\psi(x,t) = \sum_n b_n e^{-in^2\pi^2 t}\sin(n\pi x) \tag{32}$$

We now impose the initial condition $\psi(x,0) = 1$:

$$\psi(x,0) = \sum_n b_n \sin(n\pi x) \tag{33}$$

$$b_n = 2\int_0^1 \sin n\pi x\,dx = \frac{2}{n\pi}\Big[-\cos(n\pi) + 1\Big] = \frac{2[1-(-1)^n]}{n\pi} = \begin{cases} \dfrac{4}{n\pi}\,, & \text{for odd } n\,, \\ 0\,, & \text{for even } n\,. \end{cases} \tag{34}$$

Hence, the solution is:

$$\psi(x,t) = \frac{4}{\pi}\sum_{n \text{ odd}} \frac{\sin(n\pi x)}{n}\, e^{-in^2\pi^2 t}\,. \tag{35}$$