Problem 1

$$y' - 2xy = 1, \qquad y(0) = 0. \tag{1}$$

(a) Substituting $y = \sum_n a_n x^n$,

$$\sum_{n=1}^{\infty} n a_n x^{n-1} - 2\sum_{n=0}^{\infty} a_n x^{n+1} = a_1 + \sum_{n=0}^{\infty} \left[(n+2)a_{n+2} - 2a_n\right] x^{n+1} = 1, \tag{2}$$

$$a_{n+2} = \frac{2}{n+2} a_n = \frac{4}{n(n+2)} a_{n-2} = \ldots \tag{3}$$

while from $y(0) = 0$ we get $a_0 = 0$ and from the x^0 term in (2) we have $a_1 = 1$. This tells us that

$$a_{2n} = \frac{2}{2n} a_{2n-1} = \frac{4}{2n(2n-2)} a_{2n-4} = \ldots = \frac{2^n}{(2n)!!} a_0 = 0; \tag{4}$$

$$a_{2n+1} = \frac{2}{2n+1} a_{2n-1} = \frac{4}{(2n+1)(2n-1)} a_{2n-3} = \ldots = \frac{2^n}{(2n+1)!!} a_1 = \frac{2^n}{(2n+1)!!}. \tag{5}$$

Hence,

$$y(x) = \sum_{n=0}^{\infty} \frac{2^n\, x^{2n+1}}{(2n+1)!!}\,. \tag{6}$$

(b) This is a linear first order differential equation. First we solve the homogeneous equation

$$y' - 2xy = 0\,, \tag{7}$$

$$\frac{y'}{y} - 2x = 0 \implies \log y = x^2 + \log c \implies y(x) = ce^{x^2}\,. \tag{8}$$

The general solution to the inhomogeneous equation will be given by the solution to the homogeneous plus a particular solution to the inhomogeneous; the latter is given by

$$y_0(x) = e^{x^2} \int e^{-x^2} dx\,. \tag{9}$$

Recalling the definition of the error function,

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2}\, dt\,, \tag{10}$$

it follows that the general solution to the differential equation (7) is

$$y(x) = e^{x^2} \left[\frac{\sqrt{\pi}}{2} \operatorname{erf}(x) + C\right]\,, \tag{11}$$

where C is an integration constant that is determined by the initial condition $y(0) = 0$. Since $\mathrm{erf}(0) = 0$, we deduce that $C = 0$ and conclude that

$$y(x) = \frac{\sqrt{\pi}}{2}\, e^{x^2}\, \mathrm{erf}(x)\,. \tag{12}$$

One can check the series expansion. Inserting $e^{x^2} = \sum_0^\infty x^{2n}/n!$ into (10) and integrating term by term yields

$$\frac{\sqrt{\pi}}{2}\,\mathrm{erf}(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)n!}\,. \tag{13}$$

Multiplying this series with the series for e^{x^2}, one can check that the end result coincides with (6).

Problem 2

The associated Legendre functions P_n^m are defined as

$$P_n^m(x) = (1-x^2)^{m/2}\frac{d^m}{dx^m}P_n(x) \tag{14}$$

Then

$$P^1_{2n+1}(x) = \sqrt{1-x^2}\,P'_{2n+1}(x) = \sqrt{1-x^2}\cdot\frac{(2n+1)P_{2n}(x) - (2n+1)xP_{2n+1}(x)}{1-x^2} \tag{15}$$

$$P^1_{2n+1}(0) = (2n+1)P_{2n}(0) = (2n+1)\frac{(-1)^n(2n-1)!!}{2^n n!} = \frac{(-1)^n(2n+1)!!}{2^n n!} \tag{16}$$

where we have used the result for $P_{2n}(0)$ found in homework set #1, problem 12 (Boas, p. 615).

Problem 3

$$xy'' + 2y' - y = 0 \tag{17}$$

(a) Using the method of Frobenius we insert the series $y = \sum_n a_n x^{n+s}$

$$\sum_n^\infty (n+s)(n+s-1)a_n x^{n+s-1} + 2\sum_n^\infty (n+s)a_n x^{n+s-1} - \sum_n a_n x^{n+s} = 0 \tag{18}$$

The $a_0 x^{s-1}$ coefficient gives the indicial equation:

$$s(s-1) + 2s = 0 \implies s = 0, -1; \tag{19}$$

Consider first the indicial root $s = 0$. The resulting recursion relation is then given by

$$n(n+1)a_{n+1} + 2(n+1)a_{n+1} - a_n = 0 \qquad \implies a_{n+1} = \frac{a_n}{(n+2)(n+1)}\,, \tag{20}$$

which yield

$$
\begin{aligned}
c_n &= \frac{c_{n-1}}{(n+1)\cdot n} = \frac{c_{n-2}}{(n+1)\cdot n\cdot n\cdot (n-1)} \\
&= \frac{c_{n-3}}{(n+1)\cdot n\cdot n\cdot (n-1)\cdot (n-1)\cdot (n-2)} = \cdots = \frac{c_0}{(n+1)!\,n!}\,. \qquad (21)
\end{aligned}
$$

Hence, the first solution is proportional to

$$
y_1(x) = \sum_{n=0}^{\infty} \frac{x^n}{(n+1)!\,n!}\,. \qquad (22)
$$

(b) The two roots of the indicial equation differ by an integer number. If we examine the second indicial root $s = -1$, the resulting recursion relation is given by

$$
n(n-1)a_{n+1} + 2na_{n+1} - a_n = 0 \implies a_{n+1} = \frac{a_n}{n(n+1)}\,, \qquad (23)
$$

which yields that exact same series as the one obtained in part (a). This happens because the equation is Fuchsian and the two roots s_1, s_2 differ by an integer. We still have to find another linearly independent solution. Using Fuchs theorem, we write the other solution as

$$
y_2(x) = y_1(x)\,\ln x + \sum_{n=0}^{\infty} b_n x^{n-1} \qquad (24)
$$

Plugging this back into the differential equation, one can obtain a recursion relation for the b_n. Alternatively, one can make use of the class handout entitled *Series solutions to a second order linear differential equation with regular singular points.* Either method is straightforward (though somewhat tedious). The end result is:

$$
y_2(x) = y_1(x)\,\ln x + \frac{1}{x}\left[1 - \sum_{n=1}^{\infty} \frac{1}{n!\,(n-1)!}\left(\frac{1}{n} + 2\sum_{k=1}^{n-1}\frac{1}{k}\right)x^n\right]. \qquad (25)
$$

Problem 4

$$
y'' + e^{2x}y = 0 \qquad (26)
$$

If we change variables, $z = e^x$, we have

$$
\begin{aligned}
&\frac{d}{dx}y = \frac{dz}{dx}\frac{d}{dz}y(z) = zy'(z) && (27)\\
&\frac{d^2}{dx^2}y = \frac{d}{dx}y'(x) = \frac{dz}{dx}\frac{d}{dz}(zy'(z)) = z^2y''(z) + zy'(z) && (28)\\
&z^2y''(z) + zy'(z) + z^2y(z) = 0 && (29)
\end{aligned}
$$

This is Bessel's equation with $p = 0$, so the two linearly independent solutions are the Bessel functions of the first and second kind:

$$
y(x) = AJ_0(e^x) + BN_0(e^x) \qquad (30)
$$

Problem 5

Obtain a general expression for $H_n(0)$, for n non-negative integer:

We use the generating function for the Hermite polynomials:

$$\Phi(x,h) = e^{2xh-h^2} = \sum_{n=0}^{\infty} H_n(x)\frac{h^n}{n!}\,. \tag{31}$$

To find the polynomials in 0, we look at

$$\Phi(0,h) = e^{-h^2} = \sum_n H_n(0)\frac{h^n}{n!} = \sum_m \frac{(-1)^m h^{2m}}{m!} \tag{32}$$

so that

$$H_{2m}(0) = (-1)^m \frac{(2m)!}{m!}\,, \qquad H_{2m+1}(0) = 0. \tag{33}$$

Alternatively, we can use the recursion relation given by eq. (22.17)(b) on p. 609 of Boas,

$$H_{n+1}(x) = 2xH_n(x) - 2nH_{n-1}(x)\,. \tag{34}$$

Setting $x = 0$ yields,

$$H_{n+1}(0) = -2nH_{n-1}(0) = 2n\cdot(2n-2)H_{n-3}(0) = \cdots \tag{35}$$

Using the fact that $H_0(x) = 1$ and $H_1(x) = 2x$, it follows that $H_0(0) = 1$ and $H_1(0) = 0$. Clearly, (35) implies that $H_{2n+1}(0) = 0$ for any non-negative integer n. If we rewrite (35) as

$$H_{2n}(0) = -2(2n-1)H_{2n-2}(0) = (-2)^2(2n-1)(2n-3)H_{2n-4}(0) = \cdots\,, \tag{36}$$

and use $H_0(0) = 1$, then it immediately follows that

$$H_{2n}(0) = (-1)^n 2^n (2n-1)!!\,, \qquad H_{2n+1}(0) = 0. \tag{37}$$

One can check that

$$2^n(2n-1)!! = \frac{(2n)!}{n!}\,, \tag{38}$$

so that the two results above coincide as expected.

Problem 6

Let $T_n(\cos\theta) = \cos n\theta$. one can write T_n in the form

$$T_n(\cos\theta) = \sum_{m=1}^{n} a_m \cos^m\theta \tag{39}$$

The polynomials $T_n(x)$ are called Chebyshev polynomials.

(a) evaluate T_0, T_1, T_2:

$$T_0(\cos\theta) = 1 \implies T_0(x) = 1\,; \tag{40}$$
$$T_1(\cos\theta) = \cos\theta \implies T_1(x) = x\,; \tag{41}$$
$$T_2(\cos\theta) = \cos 2\theta = 2\cos^2\theta - 1 \implies T_2(x) = 2x^2 - 1. \tag{42}$$

(b) Note that $y(\theta) = T_n(\cos\theta)$ satisfy

$$y'' + n^2 y = 0 \tag{43}$$

From this we can find the equation that $T_n(x)$ satisfies:

$$x = \cos\theta\,, \qquad \frac{d}{d\theta} = \frac{dx}{d\theta}\frac{d}{dx} = -\sqrt{1-x^2}\frac{d}{dx}\,; \tag{44}$$
$$\frac{d^2}{d\theta^2} = \left(\frac{dx}{d\theta}\frac{d}{dx}\right)\left(\frac{dx}{d\theta}\frac{d}{dx}\right) = \left(\sqrt{1-x^2}\frac{d}{dx}\right)^2 = (1-x^2)\frac{d^2}{dx^2} - x\frac{d}{dx}; \tag{45}$$
$$y'' + n^2 y = 0: \tag{46}$$
$$\implies (1-x^2)T_n''(x) - xT_n'(x) + n^2 T_n(x) = 0 \tag{47}$$

(c) The differential equation for $T_n(x)$ (47) can be written as a Sturm-Liouville problem over the interval $[-1,1]$:

$$\sqrt{1-x^2}\frac{d}{dx}\left[\sqrt{1-x^2}T_n'\right] + n^2 T_n = 0 \implies \frac{d}{dx}\left[\sqrt{1-x^2}\,T_n'\right] + \frac{n^2}{\sqrt{1-x^2}}T_n = 0 \tag{48}$$

if we compare it to the standard form of a Sturm-Liouville problem, $[A(x)y']' + [\lambda B(x) + C(x)]y = 0$, we find

$$A(x) = \sqrt{1-x^2}, \qquad B(x) = \frac{1}{\sqrt{1-x^2}}\,, \qquad C(x) = 0\,, \qquad \lambda = n^2; \tag{49}$$

Moreover, the Sturm-Liouville boundary condition,

$$A(x)\left[y_n'(x)y_m(x) - y_m'(x)y_n(x)\right]\Big|_{-1}^{1} = 0\,, \tag{50}$$

is automatically satisfied when y_n is a solution to the Chebyshev differential equation with $\lambda = n^2$, since $A(\pm 1) = 0$, whereas the solutions $y_n(x)$ are non-singular at $x = \pm 1$ since they are polynomials of degree n.

(d) In this problem, $y_n(x) = T_n(x)$ is a real polynomial of degree n and the weight function is $B(x) = (1-x^2)^{-1/2}$. Hence, using the orthogonality relations proved in class for the solutions to the Sturm-Liouville problem, we conclude that the orthogonality condition satisfied by T_n is given by

$$\int_{-1}^{1} T_n(x)T_m(x)\frac{dx}{\sqrt{1-x^2}} = 0\,, \quad \text{for } n \neq m\,. \tag{51}$$

Problem 7

$$\frac{d^2y}{dx^2} + k^2 y = 0, \qquad -\ell < x < \ell\,, \qquad y(-\ell) = y(\ell) \tag{52}$$

(a) To show that the Sturm-Liouville boundary conditions are satisfied, we must make use of the eigenfunctions found in part (c) below,

$$y_n(x) = \frac{1}{\sqrt{2\ell}}\, e^{i\pi n x/\ell}\,, \quad \text{for } n = 0, \pm 1, \pm 2, \ldots \tag{53}$$

The Sturm-Liouville boundary condition for this problem is:

$$\left[y_n'(x) y_m(x) - y_m'(x) y_n(x)\right]\Big|_{-\ell}^{\ell} = 0\,, \tag{54}$$

which are clearly satisfied by the $y_n(x)$ given above, since both $y_n(x)$ and $y'(x)$ are periodic functions with period 2ℓ.

(b) The function e^{ikx} solves the equation; to satisfy the periodic boundary condition, we must have

$$e^{ik\ell} = e^{-ik\ell} \implies k = \frac{\pi}{\ell} n,\ \ n = 0, \pm 1, \pm 2, ... \tag{55}$$

(c) This set forms an orthonormal set of functions over the interval $-\ell < x < \ell$:

$$\int_{-\ell}^{\ell} e^{-i\pi n x/\ell} e^{i\pi m x/\ell} dx = \begin{cases} 0, \ \text{for } n \neq m \\ 2\ell, \ \text{for } n = m \end{cases} \tag{56}$$

The set of orthonormal functions is then $\left\{\frac{1}{\sqrt{2\ell}}\, e^{i\pi n x/\ell}, n = 0, \pm 1, \pm 2, \ldots\right\}$

(d) Given that the eigenfunctions are a complete orthogonal set, we can expand any function in terms of $\{e^{ikx}\}$: for $f(x) = x^2$, we have

$$f(x) = x^2 = \sum_{n=-\infty}^{\infty} b_n e^{in\pi x/\ell}\,, \tag{57}$$

which yields

$$\begin{aligned}
\int_{-\ell}^{\ell} e^{im\pi x/\ell} \cdot \sum_n b_n e^{in\pi x/\ell} dx = 2\ell b_m &= \int_{-\ell}^{\ell} x^2 e^{im\pi x/\ell} dx \\
&= \frac{\ell}{im\pi} x^2 e^{im\pi x/\ell}\Big|_{-\ell}^{\ell} - 2\frac{\ell}{im\pi}\int_{-\ell}^{\ell} x e^{im\pi x/\ell} dx \\
&= \frac{4\ell^3}{m^2\pi^2}(-1)^m
\end{aligned} \tag{58}$$

where the result holds for $m \neq 0$; b_0 can be calculated directly and it is $\frac{1}{3}l^2$. Finally, x^2 can be expanded in this basis in the following form:

$$f(x) = x^2 = \frac{1}{3}\ell^2 + \frac{2\ell^2}{\pi^2}\sum_{m=1}^{\infty}\frac{(-1)^m}{m^2} e^{im\pi x/\ell} \tag{59}$$

(e) If in the previous expression we set $x = \ell$, we have

$$l^2 = \frac{1}{3}l^2 + \frac{2\ell^2}{\pi^2}\sum_{n=1} \frac{1}{n^2} = \frac{1}{3}\ell^2 + \frac{4\ell^2}{\pi^2}\sum_{n=1}^{\infty} \frac{1}{n^2} \tag{60}$$

Then we can find the infinite sum

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6} \tag{61}$$

We can note that this is a particular value of the Riemann Zeta function $\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$.

Problem 8

Consider a rectangular plate $0 < x < 1$, $0 < y < 2$ with boundary condition $T(x, 2) = 0$ and $T(x, 0) = 1 - x$.

(a) Find the steady state temperature with $T(0, y) = T(1, y) = 0$.

In class, we showed that the steady state temperature distribution for the plate with $T(x, 2) = T(0, y) = T(1, y) = 0$ is given by

$$T(x, y) = \sum_{n=1}^{\infty} c_n \sinh\left[n\pi(y-2)\right] \sin(n\pi x)\,, \tag{62}$$

as this is the most general solution to the two-dimensional Laplace equation that satisfies the stated boundary conditions. To determine the c_n, we impose the final boundary condition, $T(x, 0) = 1-x$, which yields

$$1 - x = -\sinh(2n\pi) \sum_{n=1}^{\infty} c_n \sin(n\pi x)\,. \tag{63}$$

This is a Fourier sine series. The c_n are then determined by

$$c_n = \frac{-2}{\sinh(2n\pi)} \int_0^1 (1-x)\sin(n\pi x)\, dx = \frac{-2}{n\pi\, \sinh(2n\pi)}\,. \tag{64}$$

Thus,

$$T(x, y) = \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{\sinh\left[n\pi(2-y)\right]\sin(n\pi x)}{n\sinh(2n\pi)}\,. \tag{65}$$

(b) Find the steady state temperature if the sides $x = 0$, $x = 1$ are insulated.

The calculation of part (a) is modified since now we have insulated sides $x = 0$ and $x = 1$ which means that

$$\left(\frac{\partial T}{\partial x}\right)\bigg|_{x=0} = \left(\frac{\partial T}{\partial x}\right)\bigg|_{x=1} = 0\,, \tag{66}$$

which replaces the previous boundary conditions $T(0, y) = T(1, y) = 0$. Then the steady state temperature distribution for the plate is given by:

$$T(x, y) = b_0(y-2) + \sum_{n=1}^{\infty} b_n \sinh\left[n\pi(y-2)\right]\,\cos(n\pi x)\,, \tag{67}$$

as this is the most general solution to the two-dimensional Laplace equation that satisfies the stated boundary conditions. Note that we have explicitly included the $n = 0$ part of the sum. When solving Laplace's equation via separation of variables, we write $T(x, y) = X(x)Y(y)$. For the case of $n = 0$, we have $X'' = Y'' = 0$. To satisfy the boundary conditions, the solution for the $n = 0$ part of the sum must be $X(x)Y(y) = b_0(y-2)$. To determine the b_n, we impose the final boundary condition, $T(x, 0) = 1 - x$, which yields

$$1 - x = -2b_0 - \sinh(2n\pi) \sum_{n=1}^{\infty} b_n \cos(n\pi x)\,. \tag{68}$$

This is a Fourier cosine series. The b_n are then determined by

$$-\,2b_0 = \int_0^1 (1-x)dx = \tfrac{1}{2}\,, \tag{69}$$

$$b_n = \frac{-2}{\sinh(2n\pi)} \int_0^1 (1-x)\cos(n\pi x)\,dx = \frac{-2[1-(-1)^n]}{(n\pi)^2 \sinh(2n\pi)}\,, \quad \text{for } n = 1, 2, 3, \ldots. \tag{70}$$

Thus,

$$T(x, y) = \tfrac{1}{4}(2-y) + \frac{4}{\pi^2} \sum_{n\ \text{odd}} \frac{\sinh\left[n\pi(2-y)\right]\cos(n\pi x)}{n^2 \sinh(2n\pi)}\,. \tag{71}$$

Problem 9

A bar of length 2 is initially at $0°$. For time $t \geq 0$, the $x = 0$ end of the bar is insulated and the $x = 2$ end is held at $100°$. Find the time-dependent temperature distribution of the bar.

In the absence of information about the extent of the bar in the y direction, we will assume that the length in the y direction is much larger than length 2. In this case, we may neglect any end effects and assume that heat flows only in the x-direction. The boundary conditions and the initial conditions for the temperature $u(x, t)$ are:

$$u(x, 0) = 0\,, \quad u(0, t) = 0\,, \quad u(2, t) = 100\,, \tag{72}$$

and $u(x, t)$ satisfies the heat flow equation

$$\nabla^2 u = \frac{1}{\alpha^2} \frac{\partial u}{\partial t}\,. \tag{73}$$

The easiest way to solve this problem is to define a new quantity,

$$v(x, t) = u(x, t) - 50x\,. \tag{74}$$

Clearly, $v(x, t)$ also satisfies the heat flow equation, since the second space derivative and the first time derivative of the added term $-50x$ vanishes. Moreover, the boundary conditions and the initial conditions for $v(x, t)$ are:

$$v(x, 0) = -50x\,, \quad v(0, t) = 0\,, \quad v(2, t) = 0\,. \tag{75}$$

Thus, we can immediately use eq. (3.12) on p. 630 of Boas to write:

$$v(x,t)=\sum_{n=1}^{\infty} b_n e^{-(n\pi\alpha/2)^2 t}\sin\left(\frac{n\pi x}{2}\right)\,, \tag{76}$$

after putting $l=2$ for the length of the bar. Applying the initial condition $v(x,0)=-50x$ then yields

$$-50x=\sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{2}\right)\,, \tag{77}$$

This is a Fourier sine series. The coefficients b_n are then obtained from

$$b_n=\int_0^2 (-50x)\sin\left(\frac{n\pi x}{2}\right)dx\,. \tag{78}$$

Defining a new variable $y=\frac{1}{2}n\pi x$, it follows that

$$b_n=-\frac{200}{(n\pi)^2}\int_0^{n\pi} y\,\sin y\,dy=-\frac{200}{(n\pi)^2}\left(\sin y-y\cos y\right)\Big|_0^{n\pi}=\frac{200}{n\pi}(-1)^n\,. \tag{79}$$

Inserting this result into (76) and using (74), we end up with:

$$u(x,t)=50x+\frac{200}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\,e^{-(n\pi\alpha/2)^2 t}\sin\left(\frac{n\pi x}{2}\right)\,. \tag{80}$$

Problem 10

A string of length ℓ has zero initial velocity and a given displacement $y_0(x)$. Find the displacement as a function of x and t.

$$y_0(x)=\begin{cases}4hx/\ell\,, & \text{for } 0\le x\le \frac{1}{4}\ell\,,\\ 2h-4hx/\ell\,, & \text{for } \frac{1}{4}\ell\le x\le\frac{1}{2}\ell\,,\\ 0\,, & \text{for } \frac{1}{2}\ell\le x\le \ell\,,\end{cases} \tag{81}$$

with the boundary condition $y(0,t)=0,\ y'(l,t)=0$. The wave equation is

$$\frac{\partial^2 y}{\partial^2 x}=\frac{1}{v^2}\frac{\partial^2 y}{\partial^2 t}; \tag{82}$$

separating the variables $y(x,t)=X(x)T(t)$, the solution is

$$y(x,t)=\left\{\begin{array}{c}\sin kx\\ \cos kx\end{array}\right\}\times\left\{\begin{array}{c}\sin\omega t\\ \cos\omega t\end{array}\right\},\qquad \omega=kv. \tag{83}$$

Now we must apply the given boundary and initial conditions. Because $y(0,t)=0$, the cosine does not contribute. At the free end, we have $y'(\ell,t)=0$, which implies that $\cos k\ell=0$, so that $k_n=(n+\frac{1}{2})\pi$. The most general solution before applying the initial conditions are:

$$y(x,t)=\sum_{n=0}^{\infty}\sin\left[\frac{(n+\frac{1}{2})\pi x}{\ell}\right]\left\{A_n\cos\left[\frac{(n+\frac{1}{2})\pi vt}{\ell}\right]+B_n\sin\left[\frac{(n+\frac{1}{2})\pi vt}{\ell}\right]\right\} \tag{84}$$

As the string is initially not moving, $B_n = 0$. We find A_n from the initial configuration of the string,

$$y(x,0) = y_0(x) = \sum_{n=0}^{\infty} A_n \sin \frac{(n+\frac{1}{2})\pi x}{\ell} . \tag{85}$$

This is a Fourier sine series. The coefficients A_n are obtained as follows:

$$\begin{aligned} A_n &= \frac{2}{\ell} \int_0^{\ell} y_0(x) \sin \left[\frac{(n+\frac{1}{2})\pi x}{\ell} \right] \\ &= \frac{8h}{\ell^2} \int_0^{\ell/4} x \sin \left[\frac{(n+\frac{1}{2})\pi x}{\ell} \right] dx + \frac{4h}{\ell} \int_{\ell/4}^{\ell/2} \left(1 - \frac{2x}{\ell} \right) \sin \left[\frac{(n+\frac{1}{2})\pi x}{\ell} \right] dx \\ &= \frac{128 h \sin^2 \left[\frac{1}{8}(n+\frac{1}{2})\pi \right] \sin \left[\frac{1}{4}(n+\frac{1}{2})\pi \right]}{(2n+1)^2 \pi^2} . \end{aligned} \tag{86}$$

Plugging in $B_n = 0$ and A_n obtained above into (84) then yields

$$y(x,t) = \frac{128h}{\pi^2} \sum_{n=0}^{\infty} \frac{\sin^2 \left[\frac{1}{8}(n+\frac{1}{2})\pi \right] \sin \left[\frac{1}{4}(n+\frac{1}{2})\pi \right]}{(2n+1)^2} \sin \left[\frac{(n+\frac{1}{2})\pi x}{\ell} \right] \cos \left[\frac{(n+\frac{1}{2})\pi v t}{\ell} \right] . \tag{87}$$

Problem 11

A square membrane of side ℓ is distorted into the shape $f(x,y) = xy(\ell - x)(\ell - y)$ and released. Following problem 9 of homework #5, we first separate the space and time variables by writing $z(x,y,t) = F(x,y)T(t)$, which yields

$$\nabla^2 F + K^2 F = 0 \qquad\qquad \ddot{T} + K^2 v^2 T = 0 . \tag{88}$$

We now separate the variables x, y and find the solutions for a membrane fixed at its sides:

$$\frac{X''(x)}{X} + \frac{Y''(y)}{Y} + K^2 = 0 \qquad \Longrightarrow \qquad \begin{cases} X'' + k_x^2 X = 0 \\ Y'' + k_y^2 Y = 0 \end{cases} , \; k_x^2 + k_y^2 = K^2 \tag{89}$$

$$X(x)Y(y) = \sin \frac{n\pi x}{\ell} \sin \frac{m\pi y}{\ell} \tag{90}$$

$$Z(x,y,t) = \sum_{nm} \sin \frac{n\pi x}{\ell} \sin \frac{m\pi y}{\ell} \left(A_{nm} \cos \omega_{nm} t + B_{nm} \sin \omega_{nm} t \right) , \qquad \omega_{nm} = K_{nm} v, \tag{91}$$

where $K_{nm} = \sqrt{k_{x(n)}^2 + k_{y(m)}^2} = \pi \sqrt{n^2 + m^2}/\ell$. Given that the membrane is initially not moving we have $B_{nm} = 0$. Due to the form of the initial shape of the membrane, the x and y oscillations can be treated separately. Indeed, we can write $A_{nm} = A_n B_m$, where A_n and B_m are determined via the initial conditions. For example, A_n is obtained as follows:

$$A_n = \frac{2}{\ell} \int_0^{\ell} x(\ell - x) \sin \frac{n\pi x}{\ell} dx = \frac{2}{\ell} \frac{\ell^3}{\pi^3} \int_0^{\pi} y(\pi - y) \sin ny \, dy = \frac{4\ell^2}{\pi^3 n^3} \left[1 - (-1)^n \right] . \tag{92}$$

We see that only modes with odd n contribute. The computation of B_m is almost identical to the one just given, and one finds that only modes with odd m contribute. Thus, the shape of the membrane at any time $t > 0$ is then given by

$$z(x,y,t) = \frac{64\ell^4}{\pi^6} \sum_{\text{odd } n} \sum_{\text{odd } m} \frac{1}{n^3 m^3} \sin \frac{n\pi x}{\ell} \sin \frac{m\pi y}{\ell} \cos \left[\frac{\pi v (n^2 + m^2)^{1/2} t}{\ell} \right] . \tag{93}$$