1. Consider the differential equation,

$$y'' + \frac{1}{x^2}y' - \frac{c}{x^2}y = 0\,, \tag{1}$$

where c is a real number. Assume that a solution exists of the form

$$y(x) = \sum_{n=0}^{\infty} a_n x^{n+p}\,, \qquad \text{where } a_0 \neq 0\,, \tag{2}$$

where p is a number to be determined.

(a) Find the recurrence relation satisfied by the a_n.

Plugging eq. (2) into eq. (1) yields

$$\sum_{n=0}^{\infty}\left\{a_n(n+p)(n+p-1)x^{n+p-2} + a_n(n+p)x^{n+p-3} - ca_n x^{n+p-2}\right\} = 0\,.$$

We isolate the x^{p-3} term and relabel the summation index in the second term above to obtain:

$$a_0 p x^{p-3} + \sum_{n=0}^{\infty}\left\{a_n\left[(n+p)(n+p-1) - c\right] + a_{n+1}(n+p+1)\right\}x^{n+p-2} = 0\,.$$

Since $a_0 \neq 0$, it follows that $p = 0$. Thus, we are left with:

$$\sum_{n=0}^{\infty}\left\{a_n\left[n(n-1) - c\right] + a_{n+1}(n+1)\right\}x^{n-2} = 0\,.$$

Setting the coefficient of x^{n-2} to zero yields the recurrence relation,

$$a_{n+1} = \frac{[c - n(n-1)]a_n}{n+1}\,, \qquad \text{for } n = 0, 1, 2, 3, \ldots. \tag{3}$$

(b) For what values of c is $y(x)$ a polynomial of finite degree?

The solution $y(x)$ is a polynomial of a finite degree if the numerator in eq. (3) vanishes for some non-negative integer n. This occurs when $c = n(n-1)$, where n is a non-negative integer. Substituting $n = 0, 1, 2, 3, \ldots$ then yields the possible values of c,

$$c = 0, 2, 6, 12, \ldots.$$

(c) Write down the three polynomial solutions of lowest degree that satisfy eq. (1). In each case, indicate the corresponding value of c.

Without loss of generality, I can set $a_0 = 1$, since this just determines the normalization of the polynomial solution for $y(x)$. Plugging in the allowed values of c obtained in part (b) into eq. (3), we then solve for the non-zero coefficients.

If $c = 0$, then $a_n = 0$ for $n \geq 1$, in which case $y(x) = 1$. If $c = 2$, then $a_1 = 2a_0 = 2$, $a_2 = a_1$ and $a_n = 0$ for $n \geq 3$, in which case $y(x) = 1 + 2x + 2x^2$. If $c = 3$, then $a_1 = 6a_0 = 6$, $a_2 = 3a_1 = 18$, $a_3 = \frac{4}{3}a_2 = 24$, in which case $y(x) = 1 + 6x + 18x^2 + 24x^3$. One can easily check that these three polynomials are solutions to eq. (1) for the cases of $c = 0$, $c = 2$ and $c = 6$, respectively.

(d) Suppose c is not equal to any of the values obtained in part (b). In this case, the series given in eq. (2) does not terminate. Does the series for $y(x)$ converge for any nonzero x? If yes, what is the radius of convergence? If no, explain why the Frobenius method has failed.

To test for convergence, we use the ratio test [cf. eq. (6.2) on p. 13 and Section 1.10 on pp. 20–22 of Boas]. Using eq. (3)

$$\lim_{n\to\infty}\left|\frac{a_{n+1}x^{n+1}}{a_n x^n}\right| = \lim_{n\to\infty}\left|\frac{[c-n(n-1)]x}{n+1}\right| = \lim_{n\to\infty} n|x| = \infty\,,$$

for any $x \neq 0$. That is, the series solution for $y(x)$ diverges for any nonzero value of x (i.e. the radius of convergence is zero) if $y(x)$ is *not* a polynomial of finite degree. Thus, the Frobenius method has failed. The reason for this failure is that $x = 0$ is an irregular singular point of eq. (1). In particular, if we write eq. (1) in the form

$$x^2 y'' + xp(x)y' + q(x)y = 0\,,$$

then after dividing by x^2 we identify $p(x) = 1/x$ and $q(x) = -c$. Since $p(x)$ is singular at $x = 0$, we see that eq. (1) does not satisfy the Fuchsian conditions, which require that both $p(x)$ and $q(x)$ should be non-singular as $x \to 0$. Consequently, the success of the Frobenius method is not guaranteed in this case. Nevertheless, it is interesting to observe that for the special values of c obtained in part (b), the Frobenius method does correctly yield the solutions to eq. (1) that are polynomials of finite degree.

REMARK: Of course, eq. (1) must have two linearly independent solutions. Apart from the possible polynomial solutions of finite degree* when c takes on the values obtained in part (b), we expect the other solutions to possess an essential singularity at $x = 0$. An example of an essential singularity at $x = 0$ is the function $e^{1/x}$. This was the case in problem 1 of the practice final exam.

*In this case, the second linearly independent solution can be determined using the method of reduction of order [cf. case (e) on p. 434 of Boas]. I shall not present that calculation here, but I leave it to you as an exercise.

2. (a) Evaluate the following integral

$$\int_0^\infty e^{-pt} I_0(2\sqrt{at})dt\,, \qquad \text{for } p > 0\,, \tag{4}$$

by inserting the series representation for the modified Bessel function $I_0(t)$ into eq. (4) and integrating term by term. Write your final result in summation notation, where the general form of the nth term of the series is given.

The modified Bessel function is defined in eq. (17.3) on p. 595 of Boas. It therefore follows that:

$$I_0(x) = J_0(ix)\,.$$

Using the power series for $J_0(x)$ given in eq. (12.9) on p. 590 of Boas, one immediately obtains:

$$I_0(2\sqrt{at}) = \sum_{n=0}^\infty \frac{(at)^n}{[n!]^2}\,.$$

Inserting this series into eq. (4), and making use of the following integral,

$$\int_0^\infty e^{-pt} t^n\, dt = \frac{1}{p^{n+1}}\Gamma(n+1) = \frac{n!}{p^{n+1}}\,,$$

we end up with

$$\int_0^\infty e^{-pt} I_0(2\sqrt{at})dt = \sum_{n=0}^\infty \frac{a^n}{[n!]^2}\int_0^\infty e^{-pt}t^n\, dt = \sum_{n=0}^\infty \frac{a^n}{[n!]^2}\frac{n!}{p^{n+1}} = \frac{1}{p}\sum_{n=0}^\infty \frac{1}{n!}\left(\frac{a}{p}\right)^n\,.$$

(b) Sum the series and show that eq. (4) can be expressed as an elementary function.

We immediately recognize

$$\frac{1}{p}\sum_{n=0}^\infty \frac{1}{n!}\left(\frac{a}{p}\right)^n = \frac{1}{p}\, e^{a/p}\,.$$

Hence, we conclude that

$$\int_0^\infty e^{-pt} I_0(2\sqrt{at})dt = \frac{1}{p}\, e^{a/p}\,, \qquad \text{for } p > 0\,.$$

You may now add this result to your table of Laplace transforms!

3. Assume that the temperature distribution on the surface of the earth is given by:

$$T = T_2 + (T_1 - T_2)\cos^2\theta\,, \tag{5}$$

where θ is the polar angle (e.g., $\theta = \pi/2$ at the equator), T_1 is the temperature at the north and south poles, and T_2 is the temperature at the equator. Let r be the distance as measured from the center of the earth. The units of distance are chosen such that the earth is a sphere of radius 1.

(a) Show that eq. (5) can be expressed as a linear combination of two Legendre polynomials $P_\ell(x)$ (which ones?) where $x = \cos\theta$.

Using

$$P_0(\cos\theta) = 1 \quad \text{and} \quad P_2(\cos\theta) = \tfrac{1}{2}(3\cos^2\theta - 1)\,,$$

it follows that

$$\cos^2\theta = \tfrac{2}{3}P_2(\cos\theta) + \tfrac{1}{3}P_0(\cos\theta)\,.$$

Hence, eq. (5) can be rewritten as

$$\begin{aligned} T_2 + (T_1 - T_2)\cos^2\theta &= T_2 P_0(\cos\theta) + (T_1 - T_2)\left[\tfrac{2}{3}P_2(\cos\theta) + \tfrac{1}{3}P_0(\cos\theta)\right] \\ &= \tfrac{2}{3}(T_1 - T_2)P_2(\cos\theta) + \tfrac{1}{3}(T_1 + 2T_2)P_0(\cos\theta)\,. \end{aligned}$$

(b) We shall model the steady state temperature distribution of the earth by assuming that $T(r,\theta)$ is a solution to Laplace's equation in the region $0 \leq r \leq 1$, with the boundary condition $T(1,\theta)$ specified by eq. (5) [The symmetry of the problem implies that T does not depend on the azimuthal angle ϕ.] Compute $T(r,\theta)$ for all angles θ and $0 \leq r \leq 1$.

The general solution to Laplace's equation in spherical coordinates involve linear combinations of the solutions

$$T(r,\theta,\phi) = \begin{cases} r^\ell\, Y_\ell^m(\theta,\phi)\,, & \text{where } \ell = 0, 1, 2, 3, \ldots, \text{ and} \\ r^{-\ell-1}\, Y_\ell^m(\theta,\phi)\,, & m = -\ell, -\ell+1, \ldots, \ell-1, \ell\,. \end{cases}$$

We reject the solutions proportional to $r^{-\ell-1}$ since they becomes infinite as $r \to 0$. Since this problem exhibits an azimuthal symmetry, the temperature has no ϕ dependence, which means that $m = 0$. Since $Y_\ell^0(\theta,\phi)$ is proportional to the Legendre polynomial, $P_\ell(\cos\theta)$, we conclude that the general form for the solution $T(r,\theta)$ must be

$$T(r,\theta) = \sum_{\ell=0}^{\infty} c_\ell r^\ell P_\ell(\cos\theta)\,. \tag{6}$$

We now impose the boundary condition at $r = 1$. Using eq. (6) and the result of part (a), it follows that

$$T(1,\cos\theta) = \tfrac{2}{3}(T_1 - T_2)P_2(\cos\theta) + \tfrac{1}{3}(T_1 + 2T_2)P_0(\cos\theta) = \sum_{\ell=0}^{\infty} c_\ell P_\ell(\cos\theta)\,.$$

Thus, we can immediately read off the values of the coefficients c_ℓ,

$$c_0 = \tfrac{1}{3}(T_1 + 2T_2)\,, \quad c_1 = 0\,, \quad c_2 = \tfrac{2}{3}(T_1 - T_2)\,, \quad \text{and } c_\ell = 0 \text{ for } \ell \geq 3.$$

Inserting these values back into eq. (6) yields the final solution:

$$T(r,\theta) = \tfrac{1}{3}(T_1 + 2T_2) + \tfrac{1}{3}(T_1 - T_2)(3\cos^2\theta - 1)r^2\,. \tag{7}$$

(c) Evaluate $T(r,\theta)$ at the center of the earth ($r = 0$). What feature of the earth have we neglected which may be responsible for such an unrealistic final result?

Plugging $r = 0$ into eq. (7) yields

$$T(r = 0) = \tfrac{1}{3}(T_1 + 2T_2)\,,$$

for the temperature at the center of the earth. Clearly this temperature is much too small, as the temperature at the center of the earth is much much larger than any temperature at the surface. This means that eq. (7) provides a very unrealistic model for the temperature distribution inside the earth. The reason for this is that we have neglected heat sources inside the earth (which if present would be described by Poisson's equation instead of Laplace's equation). For example, natural radioactivity inside the earth's crust provides a very important source of heat.

4. In quantum mechanics, the time-independent Schrödinger equation for a free particle of energy $E = k^2$ in three dimensions is†

$$(\vec{\boldsymbol{\nabla}}^{\mathbf{2}} + k^2)\psi(\vec{\boldsymbol{r}}) = 0\,. \tag{8}$$

By convention, we take k to be positive. Suppose we confine the particle to a spherical region $0 \leq r \leq a$, where $r \equiv |\vec{\boldsymbol{r}}|$. It is convenient to employ spherical coordinates. The Laplacian can then be written as

$$\vec{\boldsymbol{\nabla}}^{\mathbf{2}} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\,. \tag{9}$$

Then, the relevant boundary conditions for this problem are:

$$\text{(i)}\quad \psi(\vec{\boldsymbol{r}}) \text{ is non-singular at the origin, and} \tag{10}$$

$$\text{(ii)}\quad \psi(a,\theta,\phi) = 0\,. \tag{11}$$

(a) Using the technique of separation of variables, write $\psi(r,\theta,\phi) = R(r)\Theta(\theta)\Phi(\phi)$, and find the ordinary differential equation that is satisfied by $R(r)$. This equation is called the *radial equation.*

†To make the algebra easier, I have taken the mass of the particle equal to $\frac{1}{2}$ and have chosen units where Planck's constant is equal to 2π.

If we perform the separation of variables following the analysis of Chapter 13, section 7 of Boas, we would insert $\psi(r,\theta,\phi) = R(r)\Theta(\theta)\Phi(\phi)$ into eq. (8), and then multiply through by $r^2/R\Theta\Phi$. The computation is nearly identical o the one given by Boas. The only change is that eq. (7.6) on p. 648 of Boas is modified to[‡]

$$\frac{1}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) + k^2r^2 = \ell(\ell+1)\,, \qquad \text{where } \ell = 0,1,2,3,\ldots. \tag{12}$$

Multiplying through by R and expanding out the derivative yields the *radial equation*:

$$r^2R'' + 2rR' + [k^2r^2 - \ell(\ell+1)]R = 0\,. \tag{13}$$

<u>*REMARK*</u>: Eq. (13) can be quickly obtained using results obtained in the class handout entitled *The Spherical Harmonics*. In the footnote that appears at the bottom of p. 4 of this handout, I noted that the Laplacian in spherical coordinates can be written as

$$\vec{\boldsymbol{\nabla}}^{\mathbf{2}} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) - \frac{\vec{\boldsymbol{L}}^{\mathbf{2}}}{r^2}\,, \tag{14}$$

where $\vec{\boldsymbol{L}}^{\mathbf{2}}$ is the differential operator,

$$\vec{\boldsymbol{L}}^{\mathbf{2}} \equiv -\frac{1}{\sin^2\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) - \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\,,$$

which depends only on the angular variables θ and ϕ. Moreover, the $Y_\ell^m(\theta,\phi)$ are eigenfunctions of the angular differential operator $\vec{\boldsymbol{L}}^{\mathbf{2}}$,

$$\vec{\boldsymbol{L}}^{\mathbf{2}} Y_\ell^m(\theta,\phi) = \ell(\ell+1)Y_\ell^m(\theta,\phi)\,. \tag{15}$$

Hence, it follows from eqs. (8) and (15) that

$$\begin{aligned}(\vec{\boldsymbol{\nabla}}^{\mathbf{2}} + k^2)\psi(\vec{\boldsymbol{r}}) &= \left\{\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) - \frac{\vec{\boldsymbol{L}}^{\mathbf{2}}}{r^2} + k^2\right\}R(r)Y_\ell^m(\theta,\phi)\\ &= \left\{\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) - \frac{\ell(\ell+1)}{r^2} + k^2)\right\}R(r)Y_\ell^m(\theta,\phi) = 0\,.\end{aligned}$$

Multiplying by $r^2/Y_\ell^m(\theta,\phi)$ then yields the radial equation given in eq. (13).

(b) What are the most general solutions to the radial equation before imposing the boundary conditions?

The radial equation obtained in part (a) is the same equation obtained in problem 7 on homework 6 [problem 13.7–16 on p. 651 of Boas]. Hence, we can immediately identify the general solution with an arbitrary linear combination of spherical Bessel functions,

$$R(r) = c_1 j_\ell(kr) + c_2 y_\ell(kr)\,. \tag{16}$$

[‡]Boas calls the separation constant k which she soon identifies as $\ell(\ell+1)$ where ℓ is any non-negative integer. Since we are using the letter k for another purpose, I have directly identified the separation constant as $\ell(\ell+1)$ in eq. (12) above.

Alternatively, you can employ eqs. (16.1) and (16.2) on p. 593 of Boas, with $a = -\frac{1}{2}$, $b = k$, $c = 1$, and $p = \ell + \frac{1}{2}$ to conclude that the most general solution to eq. (13) is

$$R(r) = \frac{1}{\sqrt{x}} \left[c_1' J_{\ell+\frac{1}{2}}(kr) + c_2' Y_{\ell+\frac{1}{2}}(kr) \right] .$$

This is equivalent to eq. (16) in light of eq. (17.4) on p. 596 of Boas, with the coefficients $c_i' = (\pi/2)^{1/2} c_i$ for $i = 1, 2$.

(c) Now, impose the relevant boundary conditions and write an implicit equation for the allowed values of k.

The relevant boundary conditions are stated in eqs. (10) and (11). Applying these to the radial function $R(r)$, we conclude that $R(0)$ is finite and $R(a) = 0$. The condition that $\lim_{r\to 0} R(r)$ is finite implies that only the spherical Bessel function $j_\ell(r)$ appears in the solution, as the functions $y_\ell(kr)$ are all singular at the origin. The condition that $R(a) = 0$ then implies that $j_\ell(ka) = 0$, which means that the allowed values of k are given by

$$k_{n\ell} = x_{n\ell}/a \,, \qquad \text{where } x_{n\ell} \text{ is the } n\text{th zero of } j_\ell(x), \text{ with } n = 1, 2, 3, \ldots.$$

(d) Using the asymptotic form for the solutions to the radial equation, write down an explicit equation for the allowed values of k (which provide a good approximation if k is large enough).

From the table on p. 604 of Boas, we see that the asymptotic behavior of $j_n(x)$ as $x \to \infty$ is given by

$$j_\ell(x) = \frac{1}{x} \sin(x - \tfrac{1}{2}\ell\pi) + \mathcal{O}(x^{-2}) \,.$$

Hence, the asymptotic form for the zeros is given by $x - \frac{1}{2}\ell\pi = n\pi$ where n is an integer, assumed large enough so that the leading term in the asymptotic expansion of $j_\ell(x)$ is dominant. That is,

$$x_{n\ell} \sim (n + \tfrac{1}{2}\ell)\pi \,.$$

Thus, we conclude that asymptotically, the allowed values of k are given by

$$k_{n\ell} \sim (n + \tfrac{1}{2}\ell)\frac{\pi}{a} \,, \qquad \text{where } n \text{ is a positive integer} \,.$$

That is, the energy levels of the system are quantized, with $E_{n\ell} = k_{n\ell}^2 = (n+\frac{1}{2}\ell)^2\pi^2/a^2$, with $n = 1, 2, 3, \ldots$ and $\ell = 0, 1, 2, 3, \ldots$.

5. In poker, each player is dealt five cards from a deck of 52 cards. The five cards dealt is called a *poker hand.* A *full house* consists of three cards of one value and two cards of another value, for example three kings and two queens. If the poker hand has four cards of one value, for example four aces (one from each of the four possible suits), then the poker hand is called a *four-of-a-kind.* Poker players sometimes wonder why four-of-a-kind beats a full house. Let's see why this is true.

(a) How many possible poker hands are there?

The number of possible poker hands is equal to the number of ways to choose five cards from a deck of 52 cards. This number is "52 choose 5," or

$$C(52,5) = \frac{52!}{47!5!} = \frac{52\cdot 51\cdot 50\cdot 49\cdot 48}{5\cdot 4\cdot 3\cdot 2\cdot 1} = 52\cdot 51\cdot 10\cdot 49\cdot 2 = 2,598,960\,.$$

(b) How many possible four-of-a-kind poker hands are there? What is the probability of being dealt a four-of-a-kind hand?

The value of the card that appears four times can take on 13 possibilities (either A, K, Q, J, 10, 9, 8, 7, 6, 5, 4, 3 or 2). Having chosen these four cards, there remain 48 possibilities for the fifth card of the poker hand. This means that the total number of possible four-of-a-kind poker hands is $13\cdot 48 = 624$. The probability that you will be dealt a four-of-a-kind is therefore

$$p(\text{four-of-a-kind}) = \frac{624}{2,598,960} = \frac{1}{4165} = 2.401\times 10^{-4}\,.$$

(c) How many possible full-house poker hands are there? What is the probability of being dealt a full house?

A full house contains three cards of one particular value and two cards of a different particular value. There are 13 possible values for the card that appears three times. Moreover, there are $C(4,3) = 4$ ways of choosing three cards of a particular value from four cards of the same value. Hence, there are $13\cdot 4 = 52$ three-card combinations in a full house. Having identified the value of the card that appears three times in a full house, this leaves 12 possible values for the card whose value appears twice in a full house. Moreover, there are $C(4,2) = 4!/(2!2!) = 6$ ways of choosing two cards of a particular value from four cards of the same value. Hence, there are $12\cdot 6 = 72$ two-card combinations in a full house. The number of possible full-house hands is equal to the product of the number of possible three-card combinations and the number of possible two-card combinations, which yields $52\cdot 72 = 3744$ possible full-house poker hands.[§]

[§] Note that it does not matter whether one starts the analysis by considering the possible three-card combinations or the two-card combinations. Had we started with the latter, we would have found $13\cdot 6 = 78$ two-card combinations and $12\cdot 4 = 48$ three-card combinations (since having chosen the value of the two-card combination, only 12 possibilities remain for the value of the three-card combination). We again find that the total number of possible full-house hands is equal to $78\cdot 48 = 3744$.

The probability that you will be dealt a full house is therefore

$$p(\text{full house}) = \frac{3744}{2,598,960} = 1.441 \times 10^{-3}\,.$$

Indeed a full house is more common that a four-of-a-kind, which explains why in poker a four-of-a-kind beats a full house.

(d) A *royal flush* is a poker hand consisting of an ace, king, queen, jack and ten of the same suit. This is the rarest of all poker hands as there are only four possible royal flush poker hands (one for each of the four suits). Suppose that two players are each dealt a poker hand. In the absence of privileged information, each player initially has the same probability of being dealt a royal flush. But suppose your opponent announces that she has a royal flush before you look at your hand. Is the probability that you have a royal flush now larger, smaller or unchanged as compared to the initial probability prior to receiving this information? Justify your response.

The probability of being dealt a royal flush is

$$p(\text{royal flush}) = \frac{4}{2,598,960} = 1.539 \times 10^{-6}\,. \tag{17}$$

Suppose that two poker hands are dealt. If you know for sure that the first hand is a royal flush, this means that in the remaining deck of 47 cards, only three possible royal flushes can be dealt. The total number of poker hands that can be dealt from a deck of 47 cards is

$$C(47,5) = \frac{47\cdot 46\cdot 45\cdot 44\cdot 43}{5\cdot 4\cdot 3\cdot 2\cdot 1} = 47\cdot 23\cdot 3\cdot 11\cdot 43 = 1,533,939\,.$$

In this case, the probability of the second royal flush, given that the first royal flush has already been dealt, is

$$p(\text{second royal flush}\,|\,\text{first royal flush was dealt}) = \frac{3}{1,533,939} = 1.956 \times 10^{-6}\,. \tag{18}$$

Remarkably, the probability of being dealt the second royal flush is now *greater* than the probability of being dealt a royal flush [given by eq. (17)] with no additional information about the other player's poker hand.

<u>*REMARK*</u>: Note that the probability above is a *conditional* probability. Thus, we could have obtained it using Bayes' formula,

$$p(\text{second royal flush}\,|\,\text{first royal flush was dealt}) = \frac{p(\text{two royal flushes are dealt})}{p(\text{first royal flush was dealt})}\,. \tag{19}$$

It is straightforward to compute the number of ways of dealing two royal flushes. There are $C(52,10)$ ways of choosing ten cards out of a deck of 52, and there are $C(10,5)$

ways of choosing two poker hands from the ten cards. There are four ways of choosing the suit for the first royal flush and three ways of choosing the suit for the second royal flush. Thus in total, there are $4 \cdot 3 = 12$ possible royal flushes from $C(52, 10) \cdot C(10, 5)$ possible pairs of poker hands. Hence,

$$p(\text{two royal flushes are dealt}) = \frac{12}{C(52,10) \cdot C(10,5)} \,.$$

Finally, we apply Bayes' formula, eq. (19), to obtain

$$\begin{aligned}
p(\text{second royal flush} \,|\, \text{first royal flush was dealt}) &= \frac{12}{C(52,10) \cdot C(10,5)} \cdot \frac{C(52,5)}{4} \\
&= \frac{3 \cdot 52!/(47! \cdot 5!)}{[52!/(42! \cdot 10!)] \cdot [10!/(5! \cdot 5!)]} \\
&= \frac{3}{[47!/(42! \cdot 5!)]} = \frac{3}{C(47,5)} \,,
\end{aligned}$$

which reproduces the result of eq. (18).

6. A continuous random variable x is said to be *uniformly distributed* over the interval $a \leq x \leq b$ (where a and b are real numbers and $a < b$) if it probability density function is given by

$$f(x) = \begin{cases} \dfrac{1}{b-a} \,, & \text{if } a \leq x \leq b \,, \\[2ex] 0 \,, & \text{if } x > b \text{ or if } x < a \,. \end{cases}$$

(a) Compute the mean of the uniformly distributed random variable x.

The mean is given by

$$\langle x \rangle = \int_{-\infty}^{\infty} x f(x) \, dx = \frac{1}{b-a} \int_a^b x \, dx = \frac{1}{b-a} \cdot \tfrac{1}{2}(b^2 - a^2) = \tfrac{1}{2}(a+b) \,. \tag{20}$$

This is to be expected. Since the continuous random variable x is uniformly distributed over the interval $a \leq x \leq b$, the expected value should be equal to the average of a and b.

(b) Compute the variance of the uniformly distributed random variable x.

To compute the variance, it is simplest to employ the formula

$$\text{Var}(x) = \langle x^2 \rangle - \langle x \rangle^2 \,.$$

The expectation value of x^2 is computed in analogy with eq. (20),

$$\langle x^2 \rangle = \int_{-\infty}^{\infty} x^2 f(x)\,dx = \frac{1}{b-a}\int_a^b x^2\,dx = \frac{1}{b-a}\cdot\tfrac{1}{3}(b^3-a^3) = \tfrac{1}{3}(a^2+ab+b^2)\,.$$

Hence,

$$\mathrm{Var}(x) = \tfrac{1}{3}(a^2+ab+b^2) - \tfrac{1}{4}(a+b)^2 = \tfrac{1}{12}(b-a)^2\,.$$

<u>*Alternative calculation*</u>: From the original definition of the variance, we obtain:

$$\mathrm{Var}(x) = \int_{-\infty}^{\infty}(x-\langle x\rangle)^2 f(x) = \frac{1}{b-a}\int_a^b \left[x-\tfrac{1}{2}(a+b)\right]^2 = \tfrac{1}{12}(b-a)^2\,,$$

as before.

(c) What is the probability that the random variable x lies within one standard deviation of the mean?

Since the standard deviation σ is defined as $\sigma = \sqrt{\mathrm{Var}(x)}$, it follows that

$$\sigma = \frac{b-a}{2\sqrt{3}}\,.$$

The probability that x lies within one standard deviation of the mean is equal to

$$p\big(\langle x\rangle - \sigma \leq x \leq \langle x\rangle + \sigma\big) = \int_{\langle x\rangle-\sigma}^{\langle x\rangle+\sigma} f(x)\,dx\,. \tag{21}$$

Since $f(x) = 0$ for $x > b$ or $x < a$, we should check to see whether $a \leq \langle x\rangle - \sigma$ and $\langle x\rangle + \sigma \leq b$. Indeed, these two inequalities are satisfied. For example, the latter requires that

$$\langle x\rangle + \sigma = \tfrac{1}{2}(a+b) + \tfrac{1}{2\sqrt{3}}(b-a) \overset{?}{\leq} b\,,$$

which can be rearranged by moving all terms proportional to b to the right-hand side of the inequality. Thus,

$$a\left(\tfrac{1}{2}-\tfrac{1}{2\sqrt{3}}\right) \overset{?}{\leq} b\left(\tfrac{1}{2}-\tfrac{1}{2\sqrt{3}}\right)\,.$$

Since $\frac{1}{2}-\frac{1}{2\sqrt{3}}$ is positive, it follows that the above inequalities are consistent with $a \leq b$. A similar analysis confirms that $a \leq \langle x\rangle - \sigma$.

Hence, eq. (21) yields

$$p\big(\langle x\rangle - \sigma \leq x \leq \langle x\rangle + \sigma\big) = \frac{1}{b-a}\int_{\langle x\rangle-\sigma}^{\langle x\rangle+\sigma} dx = \frac{2\sigma}{b-a} = \frac{1}{\sqrt{3}} = 0.5774\,.$$

That is, there is about a 58% chance that a random variable distributed uniformly will lie within one standard deviation of its mean. This should be contrasted with a normally distributed random variable, where the corresponding probability is 68%.