Here is a collection of practice problems suitable for the midterm exam.

1. Consider the differential equation:

$$y' - 2xy = 1, \qquad y(0) = 0.$$
 (1)

(a) Solve eq. (1) using the series solution technique. Writing the solution in the form $y = \sum_{n} a_n x^n$, obtain a general expression for a_n for arbitrary n. If possible, express a_n in terms of a double factorial.

(b) Solve eq. (1) by another technique and show that the solution can be written in terms of a special function studied in Physics 116A.

2. Evaluate $P_{2n+1}^1(0)$, where $P_n^m(x)$ is the associated Legendre function and n is a non-negative integer.

3. Consider the differential equation,

$$xy'' + 2y' - y = 0. (2)$$

(a) Find a generalized power series solution to eq. (2) using the method of Frobenius.

(b) Show that the second linearly independent solution to eq. (2) cannot be expressed as a generalized power series due to the presence of a logarithm. Determine the second linearly independent solution to be of the form:

$$y(x) = \ln x \sum_{n=0}^{\infty} a_n x^{n+r_1} + \sum_{n=0}^{\infty} b_n x^{n+r_2}.$$

Obtain explicitly the values of r_1 and r_2 and the general expressions for a_n and b_n for arbitrary n.

4. Find the linearly independent solutions to the differential equation,

$$y'' + e^{2x}y = 0.$$

Show that the solutions can be expressed in terms of Bessel functions.

HINT: First try a change of variables, $z = e^x$.

5. Using either the recursion relations or the generating function for the Hermite polynomials $H_n(x)$, obtain a general expression for $H_n(0)$, where n is a non-negative integer.

6. Define $T_n(\cos \theta) = \cos n\theta$. One can show that $\cos n\theta$ can be expressed as a polynomial in $\cos \theta$ of degree n.* That is, one can write $T_n(x)$ in the form:

$$T_n(x) = \sum_n a_n \cos^n \theta \,,$$

where the a_n can be obtained, for example, by the method outlined in the footnote below. For convenience, denote $x = \cos \theta$. The $T_n(x)$ are called Chebyshev polynomials.

- (a) Evaluate $T_0(x)$, $T_1(x)$ and $T_2(x)$.
- (b) Note that $y(\theta) = T_n(\cos \theta) = \cos n\theta$ satisfies

$$\frac{d^2y}{d\theta^2} + n^2y = 0.$$

Introduce a new variable $x = \cos \theta$ and let $y(x) = T_n(x)$. By appropriate use of the chain rule, find the differential equation that y(x) satisfies.

(c) Show explicitly that the differential equation for $T_n(x)$ obtained in part (b) can be written in the form of a Sturm-Liouville boundary value problem over the interval $-1 \le x \le 1$. (Make sure you verify that the Sturm-Liouville boundary condition is satisfied.)

(d) Write down the orthogonality relation satisfied by the T_n .

REMARK: The Chebyshev polynomials play a central role in the mathematical theory of the approximation of functions.

7. Consider the Sturm-Liouville problem,

$$\frac{d^2y}{dx^2} + k^2y = 0$$

for $-\ell \leq x \leq \ell$, with periodic boundary conditions,

$$y(\ell) = y(-\ell)$$

(a) Show that the Sturm-Liouville boundary condition is satisfied.

^{*}You need not offer a proof here, but in case you are curious, the proof makes use of the relations $\cos n\theta = \operatorname{Re} e^{in\theta} = \operatorname{Re} (\cos \theta + i \sin \theta)^n$, and $\sin^2 \theta = 1 - \cos^2 \theta$.

(b) Show that the eigenfunctions of the Sturm-Liouville problem can be chosen to be $\{e^{ikx}\}$, where k can only take on discrete values. Determine the allowed values of k (noting that both positive and negative values of k are allowed).

(c) Write down orthonormality relations satisfied by the eigenfunctions obtained in part (b).

(d) The eigenfunctions obtained in part (b) form a complete set. Expand the function $f(x) = x^2$ (where $-\ell \le x \le \ell$) in terms of the eigenfunctions obtained in part (b).

(e) Set $x = \ell$ in the result obtained in part (d). Show that the resulting equation provides a method for computing the infinite sum,

$$\sum_{n=1}^{\infty} \frac{1}{n^2}$$

8. Consider a rectangular plate covering the area 0 < x < 1 and 0 < y < 2. The temperature for y = 2 is T = 0 and the temperature for y = 0 is T = 1 - x.

(a) Find the steady state temperature distribution if T = 0 for x = 0 and x = 1.

(b) Find the steady state temperature distribution if the sides x = 0 and x = 1 are insulated.

9. A bar of length 2 is initially at 0°. For time $t \ge 0$, the x = 0 end of the bar is insulated and the x = 2 end is held at 100°. Find the time-dependent temperature distribution of the bar.

10. A string of length ℓ has a zero initial velocity and a displacement given by:

$$y_0(x) = \begin{cases} \frac{4hx}{\ell}, & \text{for } 0 \le x \le \frac{1}{4}\ell, \\ 2h - \frac{4h}{\ell}x, & \text{for } \frac{1}{4}\ell \le x \le \frac{1}{2}\ell, \\ 0, & \text{for } \frac{1}{2}\ell \le x \le \ell. \end{cases}$$

Assuming that the string is pinned at x = 0 and free at $x = \ell$, find the displacement of the string as a function of x and t.

11. A square membrane of side ℓ is distorted into the shape,

$$f(x,y) = xy(\ell - x)(\ell - y)$$

and released. Express its shape at subsequent times as an infinite series.