<u>FINAL EXAM ALERT:</u> On Thursday, December 8, there will be a three-hour in-class exam from 8–11 am. The exam will take place in our usual classroom (Physical Sciences Building, Room 110). During the exam, you may refer to the textbook, the class handouts, solutions to homeworks, exams and practice problems, or your own personal notes. The exam will cover all the material treated during this quarter, which includes Chapters 12, 13 and 15 of Boas and all nine homework sets. There will be a slight emphasis on the material from the second half of the quarter, although material from the first half of the quarter will be well represented.

Here is a collection of practice problems covering material from Boas sections 5, 7, 8 and 9 of Chapter 13 and all of Chapter 15 (along with the material covered on homework sets 6–9). Along with the first practice problem set, this may help you in preparing for the final exam.

1. A long cylinder has been cut into quarter cylinders that are insulated from each other. Alternate quarter cylinders are held at potentials $+100$ and -100. Find the electrostatic potential inside the cylinder.

2. Find the steady-state temperature in the region between two spheres with radii $r = 1$ and $r = 2$, respectively, if the surface of the outer sphere has its upper half held at 100° and its lower half at -100° and these temperatures are reversed for the inner sphere.

HINT: The answer should be expressed as a linear combination of two Legendre series. Find the corresponding coefficients.

3. Using the Green function technique, solve Poisson's equation,

$$\vec{\boldsymbol{\nabla}}^{\mathbf{2}}\phi(\vec{\boldsymbol{r}}) = f(\vec{\boldsymbol{r}})\,, \qquad \text{where } f(\vec{\boldsymbol{r}}) = e^{-r/a} \text{ with } r \equiv |\vec{\boldsymbol{r}}|\,, \tag{1}$$

under the assumption that $\phi(\vec{\boldsymbol{r}}) \to 0$ as $r \to \infty$. The parameter a is a constant with units of length.

HINT: Eq. (1) in homework set 7 is especially useful in evaluating the integral necessary to solve this problem.

4. Consider a metal place covering the first quadrant. The edge along the y axis is insulated and the edge along the x axis has a fixed temperature profile given by:

$$u(x,0) = \begin{cases} 100(2-x)\,, & \text{for } 0 < x < 2\,, \\ 0\,, & \text{for } x > 2\,. \end{cases}$$

Find the steady-state temperature distribution as a function of x and y. You may leave your final answer as an integral.

5. Consider the motion of a semi-infinite string with an external time-dependent force acting on it given by

$$F(t) = \cos \omega t\,, \qquad \text{for } t \geq 0\,,$$

where ω is a constant. One end of the string is kept fixed while the other end is allowed to move freely in the vertical direction. Assume that at $t = 0$, the string is initially at rest in its equilibrium position, i.e.,

$$y(x,0) = 0\,, \quad \text{and} \quad \left.\frac{\partial y(x,t)}{\partial t}\right|_{t=0} = 0\,.$$

The displacement of the string $y(x,t)$ is governed by the inhomogeneous wave equation,

$$\frac{\partial^2 y}{\partial t^2} = v^2 \frac{\partial^2 y}{\partial x^2} + F(t)\,.$$

(a) What are the relevant boundary conditions for this problem at $x = 0$ and $x = \infty$?

(b) Solve this differential equation using method of Laplace transforms. Show that the Laplace transformation of the inhomogeneous wave equation yields an ordinary differential equation. Transform the boundary conditions and then solve the resulting differential equation. Finally, apply the relevant inverse Laplace transforms to obtain the final result.

HINT: To perform the inverse Laplace transform, you will first need to apply partial fractions to the denominator of your solution. Then, consult the Laplace transform table given on pp. 469–471 of Boas. In particular, L24 and L28 will be especially useful.

6. Suppose you have two quarters and a dime in your left pocket and two dimes and three quarters in your right pocket. You select a pocket at random and from it a coin at random.

(a) What is the probability that the coin you selected is a dime?

(b) Let x be the amount of money you selected. What is the expectation value, $E(x)$?

(c) Suppose you selected in dime in part (a). What is the probability that it came from your right pocket?

(d) Suppose you do not replace the dime, but select another coin which is also a dime. What is the probability that this second coin came from your right pocket?

7. If four letters are put at random into four envelopes, what is the probability that at least one letter gets into a correct envelope?

8. A bit (i.e., a binary digit) is 0 or 1. An ordered array of eight bits (such as 01101001) is a byte.

(a) How many different bytes are there?

(b) If you select a byte at random, what is the probability that you select a byte containing three 1's and five 0's?

9. A true coin is tossed 10,000 times.

(a) Find the probability of getting exactly 5000 heads.

(b) Find the probability of getting between 4900 and 5075 heads.

Perform your calculations using both the binomial distribution and the normal approximation, and compare the results.

10. Suppose a 200-page book has, on average one misprint every ten pages. On about how many pages would you expect to find two misprints?

11. Let x_1, x_2,...,x_n be independent random variables, each with probability density function $f(x)$, mean μ and variance σ^2. Define the sample mean by

$$\overline{x} \equiv \frac{1}{n}\sum_{i=1}^{n} x_i\,.$$

Compute the expectation value $E(\overline{x})$ and the variance $\mathrm{Var}(\overline{x})$.

12. Suppose that x and y are discrete random variables, not necessarily independent.

(a) Prove that

$$E(xy) = E(x)E(y) + \mathrm{Cov}(x, y)\,,$$

where $\mathrm{Cov}(x, y)$ is covariance of x and y.

(b) A set of n measurements are made (called the "sample"), and the resulting data are (x_1, y_1), (x_2, y_2), $\ldots (x_n, y_n)$. You may assume that each measurement is independent, which implies that (x_i, y_i) and (x_j, y_j) are independent for $i \neq j$. But you cannot assume that x_i and y_i are independent. We wish to estimate the population covariance. Consider

$$V_n(x, y) = \frac{1}{n-1}\sum_{i=1}^{n}(x_i - \overline{x})(y_i - \overline{y})\,,$$

where $\overline{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$ and $\overline{y} = \frac{1}{n}\sum_{i=1}^{n} y_i$ are the corresponding sample means. Evaluate the expectation value of $V_n(x, y)$ and prove that

$$E(V_n(x, y)) = \mathrm{Cov}(x, y)\,.$$