1. A magnetic dipole $\vec{m}$ undergoes precessional motion with angular frequency $\omega$ and angle $\vartheta_0$ with respect to the $z$-axis as shown below. That is, the time-dependence of the azimuthal angle is $\varphi_0(t) = \varphi_0 - \omega t$.

Electromagnetic radiation is emitted by the precessing dipole.¹

(a) Write out an explicit expression for the time-dependent magnetic dipole vector $\vec{m}$ in terms of its magnitude $m_0$, the angles $\vartheta_0$ and $\varphi_0$ and the time $t$. Show that $\vec{m}$ consists of the sum of a time-dependent term and a time-independent term. Verify that the time-dependent term can be written as $\text{Re}(\vec{\mu} e^{-i\omega t})$, for some suitably chosen complex vector $\vec{\mu}$.

(b) Compute the angular distribution of the time-averaged radiated power, with respect to the $z$-axis defined in the above figure.

(c) Compute the total power radiated.

(d) What is the polarization of the radiation measured by an observer located along the positive $z$-axis far from the precessing dipole? How would your answer change if the observer were located in the $x$–$y$ plane?

¹Radiation from pulsars is believed to be due to this mechanism.
2. Newton’s second law in non-relativistic mechanics can be rewritten as,

\[ \vec{F} = \frac{d\vec{p}}{dt}, \]

(1)

where \( \vec{p} \) is the three-vector momentum. This is a particularly useful form, since it can be used to define the three-vector force in relativistic mechanics if \( \vec{p} \) is interpreted as the relativistic three-vector momentum.

(a) The mechanical power is defined by

\[ P = \frac{dE}{dt}, \]

where \( E \) is the relativistic energy of a particle of mass \( m \). Using the explicit expression for relativistic energy, compute its time derivative and obtain a formula for \( P \) in terms of the three-vector velocity \( \vec{v} \) and the three-vector acceleration, \( \vec{a} \equiv \frac{d\vec{v}}{dt} \).

(b) A particle of mass \( m \) moving at velocity \( \vec{v} \) is acted on by a force \( \vec{F} \). In non-relativistic mechanics, the power is equal to the rate at which work is done by the force, i.e. \( P = \vec{F} \cdot \vec{v} \). In relativistic mechanics, evaluate \( \vec{F} \cdot \vec{v} \) in terms of the three-vector velocity \( \vec{v} \) and the three-vector acceleration \( \vec{a} \), assuming that eq. (1) is the correct equation for the three-vector force. Comparing with the result of part (a), determine whether \( P = \vec{F} \cdot \vec{v} \) remains valid in relativistic mechanics. If not, determine the correct relativistic relation.

3. In a synchrotron, a curved section has a curvature radius of 10 m over an angle of 10°. An electron with a total energy of 3 GeV, moving initially in a straight line, enters the curved section. The electron subsequently exits the curved section and resumes a straight line trajectory.

(a) Compute the total energy \( E \) radiated by the electron while traversing the curved section.

\textit{HINT:} By the chain rule,

\[ \frac{dE}{dt} = \frac{dE}{d\theta} \frac{d\theta}{dt}. \]

For circular motion, one can easily relate \( d\theta/dt \) to the velocity \( v \) of the electron.

(b) Determine the wavelength and frequency of the emitted radiation corresponding to the peak of the intensity spectrum. Identify the characteristic of the emitted radiation (radio? visible light? X-ray? gamma ray? ...).

\textit{HINT:} Recall that for the electron, \( mc^2 = 511 \text{ keV} \), where 1 eV = \( 1.6 \times 10^{-12} \) ergs, and the fundamental unit of electric charge in gaussian units is \( e = 4.8 \times 10^{-10} \) esu (note that an esu is sometimes called a statcoulomb).
4. Consider the scattering of an electromagnetic wave of (angular) frequency $\omega$ and polarization $\hat{\epsilon}_0$ off of an electron bound in an atom. The wavelength of the incoming wave is assumed to be significantly larger than the size of the atom. You may also assume that the non-relativistic limit is a good approximation.

For simplicity, we will model the electron by assuming that it is bound by a damped harmonic oscillator force:

$$\vec{F} = -m\omega_0^2\vec{x}(t) - \eta m \frac{d\vec{x}(t)}{dt},$$

where $\vec{x}(t)$ is the position (at time $t$) of the electron, $\omega_0$ is the natural frequency of the oscillator, and $\eta$ is the damping coefficient. Note that the minus signs above indicate that this is a restoring force. The electron, with mass $m$ and charge $-e$, also experiences a force due to the electric field of the incoming wave.

Using Newton’s second law, it is straightforward to show that the response of the electron to the initial wave can be modeled by a time-dependent electric dipole moment,

$$\vec{p}(t) = \text{Re}(\vec{p} e^{-i\omega t}),$$

where the complex vector $\vec{p}$ is given by

$$\vec{p} = \frac{e^2}{m} \left( \frac{E_0 \hat{\epsilon}_0}{\omega_0^2 - \omega^2 - i\eta \omega} \right).$$

(a) Assuming that the incoming wave is left circularly polarized and propagates in the positive $z$-direction, compute the angular distribution of the scattering cross section, under the assumption that the final state polarization $\hat{\epsilon}$ is not observed. Express the coefficient of the angular factor as a function of the frequency of the incoming wave. You may use eq. (2) as your starting point.

(b) Integrate the differential cross section over angles to obtain the total scattering cross section. Compare your result to the Thomson cross section in three limiting cases: (i) $\omega \gg \omega_0 \sim \eta$; (ii) $\omega = \omega_0 \gg \eta$; and (iii) $\omega \ll \omega_0 \sim \eta$.

(c) [EXTRA CREDIT] Derive eq. (2).