The tensor spherical harmonics

1 The Clebsch-Gordon coefficients

Consider a system with orbital angular momentum $\vec{\boldsymbol{L}}$ and spin angular momentum $\vec{\boldsymbol{S}}$. The total angular momentum of the system is denoted by $\vec{\boldsymbol{J}} = \vec{\boldsymbol{L}} + \vec{\boldsymbol{S}}$. Clebsch Gordon coefficients allow us to express the total angular momentum basis $|j\,m\,;\,\ell\,s\rangle$ in terms of the direct product basis, $|\ell\,m_\ell\,;\,s\,m_s\rangle \equiv |\ell\,m_\ell\rangle \otimes |s\,m_s\rangle$,

$$|j\,m\,;\,\ell\,s\rangle = \sum_{m_\ell=-\ell}^{\ell}\sum_{m_s=-s}^{s} \langle \ell\,m_\ell\,;\,s\,m_s\,|\,j\,m\,;\,\ell\,s\rangle |\ell\,m_\ell\,;\,s\,m_s\rangle\,. \tag{1}$$

The Clebsch-Gordon coefficient is often denoted by (cf. pp. 412–415 of Ref. [1]):

$$\langle \ell\,m_\ell\,;\,s\,m_s\,|\,j\,m\rangle \equiv \langle \ell\,m_\ell\,;\,s\,m_s\,|\,j\,m\,;\,\ell\,s\rangle\,,$$

since including $\ell\,s$ in $|j\,m\,;\,\ell\,s\rangle$ on the right hand side above is redundant information.

One important property of the Clebsch-Gordon coefficients is

$$\langle \ell\,m_\ell\,;\,s\,m_s\,|\,j\,m\rangle = \delta_{m\,,\,m_\ell+m_s}\langle \ell\,m_\ell\,;\,s\;m_s\,|\,j\;m_\ell+m_s\rangle\,, \tag{2}$$

which implies that if $m \neq m_\ell + m_s$ then the corresponding Clebsch-Gordon coefficient must vanish. This is simply a consequence of $J_z = L_z + S_z$. Likewise, $|\ell - s| \leq j \leq \ell + s$ (where $2j$, ℓ and $2s$ are non-negative integers), otherwise the corresponding Clebsch-Gordon coefficients vanish.

Recall that in the coordinate representation, the angular moment operator is a differential operator given by

$$\vec{\boldsymbol{L}} = -i\hbar\,\vec{\boldsymbol{x}} \times \vec{\boldsymbol{\nabla}}\,.$$

The spherical harmonics, $Y_{\ell m_\ell}(\theta,\phi)$ are simultaneous eigenstates of $\vec{\boldsymbol{L}}^2$ and L_z,

$$\vec{\boldsymbol{L}}^2\,Y_{\ell m_\ell}(\theta,\phi) = \hbar^2\ell(\ell+1)\,Y_{\ell m_\ell}(\theta,\phi)\,, \qquad L_z\,Y_{\ell m_\ell}(\theta,\phi) = \hbar m_\ell\,Y_{\ell m_\ell}(\theta,\phi)\,.$$

We can generalize these results to systems with non-zero spin. First, we define $\chi_{s\,m_s}$ to be the simultaneous eigenstates of $\vec{\boldsymbol{S}}^2$ and S_z,

$$\vec{\boldsymbol{S}}^2\chi_{s\,m_s} = \hbar^2 s(s+1)\chi_{s\,m_s}\,, \qquad S_z\chi_{s\,m_s} = \hbar m_s\chi_{s\,m_s}\,.$$

The direct product basis in the coordinate representation is given by $Y_{\ell m_\ell}(\theta,\phi)\chi_{s\,m_s}$.

2 Definition of the tensor spherical harmonics

In the coordinate representation, the total angular momentum basis consists of simultaneous eigenstates of $\vec{\boldsymbol{J}}^2$, J_z , $\vec{\boldsymbol{L}}^2$, $\vec{\boldsymbol{S}}^2$. These are the *tensor spherical harmonics*, which satisfy,

$$\vec{\boldsymbol{J}}^2\, \mathcal{Y}^{\ell s}_{jm}(\theta,\phi) = \hbar^2 j(j+1)\, \mathcal{Y}^{\ell s}_{jm}(\theta,\phi)\,, \qquad J_z\, \mathcal{Y}^{\ell s}_{jm}(\theta,\phi) = \hbar m\, \mathcal{Y}^{\ell s}_{jm}(\theta,\phi)\,,$$
$$\vec{\boldsymbol{L}}^2\, \mathcal{Y}^{\ell s}_{jm}(\theta,\phi) = \hbar^2 \ell(\ell+1)\, \mathcal{Y}^{\ell s}_{jm}(\theta,\phi)\,, \qquad \vec{\boldsymbol{S}}^2\, \mathcal{Y}^{\ell s}_{jm}(\theta,\phi) = \hbar^2 s(s+1)\, \mathcal{Y}^{\ell s}_{jm}(\theta,\phi)\,.$$

As a consequence of eq. (1), the tensor spherical harmonics are defined by

$$\begin{aligned}\mathcal{Y}^{\ell s}_{jm}(\theta,\phi) &= \sum_{m_\ell=-\ell}^{\ell} \sum_{m_s=-s}^{s} \langle \ell\, m_\ell\,;\, s\, m_s\,|\, j\, m\rangle Y_{\ell m_\ell}(\theta,\phi)\chi_{s\, m_s} \\ &= \sum_{m_s=-s}^{s} \langle \ell\,,\, m-m_s\,;\, s\, m_s\,|\, j\, m\rangle Y_{\ell\,,\, m-m_s}(\theta,\phi)\chi_{s\, m_s}\,, \qquad (3)\end{aligned}$$

where the second line follows from the first line above since the Clebsch-Gordon coefficient above vanishes unless $m = m_\ell + m_s$.

The general expressions for the Clebsch-Gordon coefficients in terms of j, m_ℓ, ℓ, s and m_s are very complicated to write down. Nevertheless, the explicit expressions in the simplest cases of $s = 1/2$ and $s = 1$ are manageable. Thus, we shall exhibit these two special cases below.

3 The spinor spherical harmonics

For spin $s = 1/2$, the possible values of j are $j = \ell + \frac{1}{2}$ and $\ell - \frac{1}{2}$, for $\ell = 1, 2, 3, \ldots$. If $\ell = 0$ then only $j = \frac{1}{2}$ is possible (and the last row of Table 1 should be omitted). The corresponding table of Clebsch-Gordon coefficients is exhibited in Table 1.

Table 1: the Clebsch-Gordon coefficients, $\langle \ell\, m - m_s\,;\, \frac{1}{2}\, m_s\,|\, jm\rangle$.

j	$m_s = \frac{1}{2}$	$m_s = -\frac{1}{2}$
$\ell + \frac{1}{2}$	$\left(\dfrac{\ell + m + \frac{1}{2}}{2\ell+1}\right)^{1/2}$	$\left(\dfrac{\ell - m + \frac{1}{2}}{2\ell+1}\right)^{1/2}$
$\ell - \frac{1}{2}$	$-\left(\dfrac{\ell - m + \frac{1}{2}}{2\ell+1}\right)^{1/2}$	$\left(\dfrac{\ell + m + \frac{1}{2}}{2\ell+1}\right)^{1/2}$

Comparing with eq. (3), the entries in Table 1 are equivalent to the following result:

$$\left|j = \ell \pm \tfrac{1}{2}\, m\right\rangle = \frac{1}{\sqrt{2\ell+1}}\left[\pm\sqrt{\ell + \tfrac{1}{2} \pm m}\,\left|\ell\, m - \tfrac{1}{2}\,;\, \tfrac{1}{2}\, \tfrac{1}{2}\right\rangle + \sqrt{\ell + \tfrac{1}{2} \mp m}\,\left|\ell\, m + \tfrac{1}{2}\,;\, \tfrac{1}{2}\ -\tfrac{1}{2}\right\rangle\right].$$

We can represent $|\frac{1}{2}\,\frac{1}{2}\rangle = \binom{1}{0}$ and $|\frac{1}{2}\,-\frac{1}{2}\rangle = \binom{0}{1}$. Then in the coordinate representation, the *spin spherical harmonics* are given by

$$\mathcal{Y}^{\ell\,\frac{1}{2}}_{j=\ell\pm\frac{1}{2}\,,\,m}(\theta,\phi) \equiv \langle\theta\,\phi\,|\,j=\ell\pm\tfrac{1}{2}\,,\,m\rangle = \frac{1}{\sqrt{2\ell+1}}\begin{pmatrix} \pm\sqrt{\ell\pm m+\frac{1}{2}}\;Y_{\ell,m-\frac{1}{2}}(\theta,\phi) \\ \sqrt{\ell\mp m+\frac{1}{2}}\;Y_{\ell,m+\frac{1}{2}}(\theta,\phi)\end{pmatrix}. \qquad (4)$$

If $\ell = 0$, there is only one spin spherical harmonic,

$$\mathcal{Y}^{0\,\frac{1}{2}}_{j=\frac{1}{2}\,,\,m}(\theta,\phi) \equiv \langle\theta\,\phi\,|\,j=\tfrac{1}{2}\,,\,m\rangle = \frac{1}{\sqrt{2\ell+1}}\begin{pmatrix} \sqrt{\frac{1}{2}+m}\;Y_{0,m-\frac{1}{2}}(\theta,\phi) \\ \sqrt{\frac{1}{2}-m}\;Y_{0,m+\frac{1}{2}}(\theta,\phi)\end{pmatrix}. \qquad (5)$$

Note that when $m = \frac{1}{2}$ the lower component of eq. (5) vanishes and when $m = -\frac{1}{2}$ the upper component of eq. (5) vanishes. In both cases, the non-vanishing component is proportional to $Y_{00}(\theta,\phi) = 1/\sqrt{4\pi}$.

4 The vector spherical harmonics

For spin $s = 1$, the possible values of j are $j = \ell+1\,,\,\ell\,,\,\ell-1$ for $\ell = 1, 2, 3, \ldots$. If $\ell = 0$ then only $j = 1$ is possible (and the last two rows exhibited in Table 2 should be omitted). The corresponding table of Clebsch-Gordon coefficients is exhibited in Table 2.

Table 2: the Clebsch-Gordon coefficients, $\langle \ell\, m - m_s\,;\,1\, m_s\,|\,jm\rangle$.

j	$m_s = 1$	$m_s = 0$	$m_s = -1$
$\ell+1$	$\left[\frac{(\ell+m)(\ell+m+1)}{(2\ell+1)(2\ell+2)}\right]^{1/2}$	$\left[\frac{(\ell-m+1)(\ell+m+1)}{(\ell+1)(2\ell+1)}\right]^{1/2}$	$\left[\frac{(\ell-m)(\ell-m+1)}{(2\ell+1)(2\ell+2)}\right]^{1/2}$
ℓ	$-\left[\frac{(\ell-m+1)(\ell+m)}{2\ell(\ell+1)}\right]^{1/2}$	$\frac{m}{\sqrt{\ell(\ell+1)}}$	$\left[\frac{(\ell-m)(\ell+m+1)}{2\ell(\ell+1)}\right]^{1/2}$
$\ell-1$	$\left[\frac{(\ell-m)(\ell-m+1)}{2\ell(2\ell+1)}\right]^{1/2}$	$-\left[\frac{(\ell-m)(\ell+m)}{\ell(2\ell+1)}\right]^{1/2}$	$\left[\frac{(\ell+m)(\ell+m+1)}{2\ell(2\ell+1)}\right]^{1/2}$

Using a spherical basis, we can represent $|1\,1\rangle = \left(\begin{smallmatrix}1\\0\\0\end{smallmatrix}\right)$, $|1\,0\rangle = \left(\begin{smallmatrix}0\\1\\0\end{smallmatrix}\right)$ and $|1\,-1\rangle = \left(\begin{smallmatrix}0\\0\\1\end{smallmatrix}\right)$. With respect to this basis, we can explicitly write out the three *vector spherical harmonics*, $\mathcal{Y}^{\ell\,1}_{j=\ell\pm1\,,\,m}(\theta,\phi)$ and $\mathcal{Y}^{\ell\,1}_{j=\ell\,,\,m}(\theta,\phi)$. For example, if $\ell\neq 0$ then,

$$\mathcal{Y}^{\ell\,1}_{j=\ell\,,\,m}(\theta,\phi) = \begin{pmatrix} -\left[\dfrac{(\ell-m+1)(\ell+m)}{2\ell(\ell+1)}\right]^{1/2} Y_{\ell,m-1}(\theta,\phi) \\ \dfrac{m}{\sqrt{\ell(\ell+1)}}\, Y_{\ell m}(\theta,\phi) \\ \left[\dfrac{(\ell+m+1)(\ell-m)}{2\ell(\ell+1)}\right]^{1/2} Y_{\ell,m-1}(\theta,\phi) \end{pmatrix}.$$

The other two vector spherical harmonics can be written out in a similar fashion. If $\ell = 0$ then $\mathcal{Y}^{\ell\,1}_{j=\ell+1\,,\,m}(\theta,\phi)$ is the only surviving vector spherical harmonic.

It is instructive to work in a Cartesian basis, where the χ_{1,m_s} are eigenvectors of $\mathcal{S}_3$, and the spin-1 spin matrices are given by $\hbar\vec{\boldsymbol{\mathcal{S}}}$, where $(\mathcal{S}_k)_{ij} = -i\epsilon_{ijk}$. In particular,

$$\mathcal{S}_3 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

and $\mathcal{S}_3\chi_{1,m_s} = m_s\chi_{1,m_s}$. This yields the orthonormal eigenvectors,

$$\chi_{1,\pm1} = \frac{1}{\sqrt{2}}\begin{pmatrix} \mp1 \\ -i \\ 0 \end{pmatrix}, \qquad \chi_{1,0} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}. \tag{6}$$

where the arbitrary overall phase factors are conventionally chosen to be unity. As an example, in the Cartesian basis,

$$\mathcal{Y}^{\ell\,1}_{j=\ell\,,\,m}(\theta,\phi) = \frac{1}{2\sqrt{\ell(\ell+1)}}\begin{pmatrix} [(\ell-m+1)(\ell+m)]^{1/2}\,Y_{\ell,m-1}(\theta,\phi) + [(\ell+m+1)(\ell-m)]^{1/2}\,Y_{\ell,m+1}(\theta,\phi) \\ i\,[(\ell-m+1)(\ell+m)]^{1/2}\,Y_{\ell,m-1}(\theta,\phi) - i\,[(\ell+m+1)(\ell-m)]^{1/2}\,Y_{\ell,m+1}(\theta,\phi) \\ 2mY_{\ell m}(\theta,\phi) \end{pmatrix}. \tag{7}$$

This is a vector with respect to the basis $\{\hat{\boldsymbol{x}}\,,\,\hat{\boldsymbol{y}}\,,\,\hat{\boldsymbol{z}}\}$. It is convenient to rewrite eq. (7) in terms of the basis $\{\hat{\boldsymbol{r}}\,,\,\hat{\boldsymbol{\theta}}\,,\,\hat{\boldsymbol{\phi}}\}$ using

$$\begin{aligned} \hat{\boldsymbol{x}} &= \hat{\boldsymbol{r}}\sin\theta\cos\phi + \hat{\boldsymbol{\theta}}\cos\theta\cos\phi - \hat{\boldsymbol{\phi}}\sin\phi\,, \\ \hat{\boldsymbol{y}} &= \hat{\boldsymbol{r}}\sin\theta\sin\phi + \hat{\boldsymbol{\theta}}\cos\theta\sin\phi + \hat{\boldsymbol{\phi}}\cos\phi\,, \\ \hat{\boldsymbol{z}} &= \hat{\boldsymbol{r}}\cos\theta - \hat{\boldsymbol{\theta}}\sin\theta\,. \end{aligned}$$

We can then greatly simplify the resulting expression for $\mathcal{Y}^{\ell\,1}_{j=\ell\,,\,m}(\theta,\phi)$ by employing the recursion relation,

$$-2m\cos\theta\,Y_{\ell m}(\theta,\phi) = \sin\theta\Big\{[(\ell+m+1)(\ell-m)]^{1/2}\,e^{-i\phi}\,Y_{\ell,m+1}(\theta,\phi) + [(\ell-m+1)(\ell+m)]^{1/2}\,e^{i\phi}\,Y_{\ell,m-1}(\theta,\phi)\Big\},$$

and the following two differential relations,

$$\frac{\partial}{\partial\phi}\,Y_{\ell m}(\theta,\phi)=imY_{\ell m}(\theta,\phi)\,,$$

$$\frac{\partial}{\partial\theta}\,Y_{\ell m}(\theta,\phi)=\tfrac{1}{2}\left[(\ell+m+1)(\ell-m)\right]^{1/2}e^{-i\phi}\,Y_{\ell,m+1}(\theta,\phi)$$
$$-\tfrac{1}{2}\left[(\ell-m+1)(\ell+m)\right]^{1/2}e^{i\phi}\,Y_{\ell,m-1}(\theta,\phi)\,.$$

Following a straightforward but tedious computation, the end result is:

$$\mathcal{Y}^{\ell\,1}_{j=\ell\,,\,m}(\theta,\phi)=\frac{i}{\sqrt{\ell(\ell+1)}}\left[\frac{\hat{\boldsymbol{\theta}}}{\sin\theta}\,\frac{\partial}{\partial\phi}\,Y_{\ell m}(\theta,\phi)-\hat{\boldsymbol{\phi}}\,\frac{\partial}{\partial\theta}\,Y_{\ell m}(\theta,\phi)\right].$$

At this point, one should recognize the differential operator $\vec{\boldsymbol{L}}$ expressed in the $\{\hat{\boldsymbol{r}}\,,\,\hat{\boldsymbol{\theta}}\,,\,\hat{\boldsymbol{\phi}}\}$ basis,

$$\vec{\boldsymbol{L}}=-i\hbar\,\vec{\boldsymbol{x}}\times\vec{\boldsymbol{\nabla}}=i\hbar\left[\frac{\hat{\boldsymbol{\theta}}}{\sin\theta}\,\frac{\partial}{\partial\phi}-\hat{\boldsymbol{\phi}}\,\frac{\partial}{\partial\theta}\right].$$

Hence, we end up with

$$\mathcal{Y}^{\ell\,1}_{j=\ell\,,\,m}(\theta,\phi)=\frac{1}{\sqrt{\hbar^2\ell(\ell+1)}}\,\vec{\boldsymbol{L}}\,Y_{\ell m}(\theta,\phi)\,,\qquad\text{for }\ell\neq 0\,.\tag{8}$$

This is the vector spherical harmonic,

$$\vec{\boldsymbol{X}}_{\ell m}(\theta,\phi)=\frac{-i}{\sqrt{\ell(\ell+1)}}\,\vec{\boldsymbol{x}}\times\vec{\boldsymbol{\nabla}}\,Y_{\ell m}(\theta,\phi)\,,$$

employed by J.D. Jackson, *Classical Electrodynamics*, 3rd Edition (John Wiley & Sons, Inc., New York, 1999).

Using the same methods, one can derive the following expressions for the other two vector spherical harmonics,

$$\mathcal{Y}^{\ell\,1}_{j=\ell-1\,,\,m}(\theta,\phi)=\frac{-1}{\sqrt{(j+1)(2j+1)}}\left[(j+1)\hat{\boldsymbol{n}}-r\vec{\boldsymbol{\nabla}}\right]Y_{jm}(\theta,\phi)\,,\qquad\text{for }\ell\neq 0\,,\tag{9}$$

$$\mathcal{Y}^{\ell\,1}_{j=\ell+1\,,\,m}(\theta,\phi)=\frac{1}{\sqrt{j(2j+1)}}\left[j\hat{\boldsymbol{n}}+r\vec{\boldsymbol{\nabla}}\right]Y_{jm}(\theta,\phi)\,,\tag{10}$$

where $\vec{\boldsymbol{x}}=r\vec{\boldsymbol{n}}$ and $\hat{\boldsymbol{n}}\equiv\hat{\boldsymbol{r}}$. That is, the three independent normalized vector spherical harmonics can be chosen as:

$$\left\{\frac{-ir}{\sqrt{j(j+1)}}\,\hat{\boldsymbol{n}}\times\vec{\boldsymbol{\nabla}}\,Y_{jm}(\theta,\phi)\,,\quad\frac{r}{\sqrt{j(j+1)}}\,\vec{\boldsymbol{\nabla}}\,Y_{jm}(\theta,\phi)\,,\quad\hat{\boldsymbol{n}}\,Y_{jm}(\theta,\phi)\right\}.\tag{11}$$

It is often convenient to rewrite

$$r\,\vec{\boldsymbol{\nabla}}\;Y_{jm}(\theta,\phi)=-r[\hat{\boldsymbol{n}}(\hat{\boldsymbol{n}}\cdot\vec{\boldsymbol{\nabla}})-\vec{\boldsymbol{\nabla}}]Y_{jm}(\theta,\phi)=-r\,\hat{\boldsymbol{n}}\times(\hat{\boldsymbol{n}}\times\vec{\boldsymbol{\nabla}})Y_{jm}(\theta,\phi)\,,$$

after noting that

$$\hat{\boldsymbol{n}} \cdot \vec{\boldsymbol{\nabla}}\, Y_{jm}(\theta,\phi) = \frac{\partial Y_{jm}(\theta,\phi)}{\partial r} = 0\,.$$

Then, the list of the three independent normalized vector spherical harmonics takes the following form:

$$\left\{ \frac{-ir}{\sqrt{j(j+1)}}\, \hat{\boldsymbol{n}} \times \vec{\boldsymbol{\nabla}}\, Y_{jm}(\theta,\phi)\,, \quad \frac{-r}{\sqrt{j(j+1)}}\, \hat{\boldsymbol{n}} \times (\hat{\boldsymbol{n}} \times \vec{\boldsymbol{\nabla}}) Y_{jm}(\theta,\phi)\,, \quad \hat{\boldsymbol{n}}\, Y_{jm}(\theta,\phi) \right\}. \tag{12}$$

In particular, the first two vector spherical harmonics listed in eq. (12) are transverse (i.e., perpendicular to $\hat{\boldsymbol{n}}$), whereas the third vector spherical harmonic in eq. (12) is longitudinal (i.e., parallel to $\hat{\boldsymbol{n}}$). This is convenient for the multipole expansion of the transverse electric and magnetic radiation fields, where only the first two vector spherical harmonics of eq. (11) appear. However, the second and third vector spherical harmonics listed in eq. (11) [or eq. (12)], $r\vec{\boldsymbol{\nabla}}\, Y_{jm}(\theta,\phi)$ and $\hat{\boldsymbol{n}}\, Y_{jm}(\theta,\phi)$, are not eigenstates of $\vec{\boldsymbol{L}}^2$ since they consist of linear combinations of states with $\ell = j \pm 1$ [which can be explicitly derived by inverting eqs. (9) and (10)].

The algebraic steps involved in establishing eqs. (8)–(10) are straightforward but tedious. A more streamlined approach to the derivation of these results is given in the next section.

5 The vector spherical harmonics revisited

Since $Y_{\ell m}(\hat{\boldsymbol{n}})$ is a spherical tensor of rank-ℓ, and $\hat{\boldsymbol{n}} \equiv \vec{\boldsymbol{x}}/r$, $\vec{\boldsymbol{L}} \equiv -i\hbar\vec{\boldsymbol{x}} \times \vec{\boldsymbol{\nabla}}$ and $r\vec{\boldsymbol{\nabla}}$ are vector operators, it is not surprising that the vector spherical harmonics are linear combinations of the quantities given in eq. (11). It is instructive to derive this result directly. For convenience, we denote the vector spherical harmonics in this section by

$$\vec{\boldsymbol{Y}}_{j\ell m}(\hat{\boldsymbol{n}}) \equiv \mathcal{Y}^{\ell\,1}_{j\,m}(\theta,\phi)\,, \quad \text{for } j = \ell+1\,,\ \ell\,,\ \ell-1\,, \tag{13}$$

where $\hat{\boldsymbol{n}}$ is a unit vector with polar angle θ and azimuthal angle ϕ.

First, we recall that (see, e.g., eq. (12.5.20) of Ref. [1]):

$$L_\pm|\ell\, m\rangle = \hbar\left[(\ell \mp m)(\ell \pm m+1)\right]^{1/2} |j\, m\pm 1\rangle\,, \qquad L_z|\ell\, m\rangle = \hbar m|\ell\, m\rangle\,, \tag{14}$$

where $L_\pm \equiv L_x \pm iL_y$. The spherical components of $\vec{\boldsymbol{L}}$ are L_q $(q = +1, 0, -1)$ where

$$L_{\pm 1} \equiv \pm\frac{L_\pm}{\sqrt{2}} = \frac{1}{\sqrt{2}}\left(L_x \pm iL_y\right)\,, \qquad L_0 \equiv L_z\,.$$

Using the Clebsch-Gordon coefficients given in Table 2, it follows that

$$L_q|\ell\, m\rangle = \hbar(-1)^q\sqrt{\ell(\ell+1)}\,\langle \ell\,,\, m+q\,;\, 1\,,\, -q\,|\,\ell\, m\rangle\, |\ell\,,\, m+q\rangle\,. \tag{15}$$

In the coordinate representation, eq. (15) is equivalent to

$$L_q\, Y_{\ell m}(\hat{\boldsymbol{n}}) = \hbar(-1)^q\sqrt{\ell(\ell+1)}\,\langle \ell\,,\, m+q\,;\, 1\,,\, -q\,|\,\ell\, m\rangle\, Y_{\ell,\,m+q}(\hat{\boldsymbol{n}})\,. \tag{16}$$

It is convenient to introduce a set of spherical basis vectors,

$$\hat{\boldsymbol{e}}_{\pm 1} \equiv \mp \frac{1}{\sqrt{2}} \left(\hat{\boldsymbol{x}} \pm i \hat{\boldsymbol{y}} \right) , \qquad \hat{\boldsymbol{e}}_0 \equiv \hat{\boldsymbol{z}} \, . \tag{17}$$

It is not surprising that $\hat{\boldsymbol{e}}_q = \chi_{1,q}$ [cf. eq. (6)]. One can check that

$$\vec{\boldsymbol{L}} = L_x \hat{\boldsymbol{x}} + L_y \hat{\boldsymbol{y}} + L_z \hat{\boldsymbol{z}} = \sum_q (-1)^q L_q \hat{\boldsymbol{e}}_{-q} \, , \tag{18}$$

where the sum over q runs over $q = -1, 0, +1$.[1] Hence, eqs. (16) and (18) yield

$$\vec{\boldsymbol{L}} Y_{\ell m}(\hat{\boldsymbol{n}}) = \hbar \sqrt{\ell(\ell+1)} \sum_q \hat{\boldsymbol{e}}_{-q} \langle \ell \, , \, m+q \, ; \, 1 \, , \, -q \, | \, \ell \, m \rangle \, Y_{\ell \, , \, m+q}(\hat{\boldsymbol{n}}) \, .$$

Since the sum is taken over $q = -1, 0, 1$, we are free to relabel $q \to -q$. Writing $\hat{\boldsymbol{e}}_q = \chi_{1,q}$, we end up with

$$\vec{\boldsymbol{L}} Y_{\ell m}(\hat{\boldsymbol{n}}) = \hbar \sqrt{\ell(\ell+1)} \sum_q \langle \ell \, , \, m-q \, ; \, 1 \, , \, q \, | \, \ell \, m \rangle \, Y_{\ell \, , \, m-q}(\hat{\boldsymbol{n}}) \chi_{1 \, q} \, .$$

Comparing with eq. (3) for $s = 1$, it follows that [in the notation of eq. (13)]:

$$\boxed{\vec{\boldsymbol{L}} Y_{\ell m}(\hat{\boldsymbol{n}}) = \hbar \sqrt{\ell(\ell+1)} \, \vec{\boldsymbol{Y}}_{\ell \ell m}(\hat{\boldsymbol{n}})} \tag{19}$$

in agreement with eq. (8).

Next, we examine $\hat{\boldsymbol{n}} Y_{\ell m}(\hat{\boldsymbol{n}})$. It is convenient to expand $\hat{\boldsymbol{n}} \equiv \vec{\boldsymbol{x}}/r$ in a spherical basis. Using eq. (17), the following expression is an identity,

$$\hat{\boldsymbol{n}} = \hat{\boldsymbol{x}} \sin\theta \cos\phi + \hat{\boldsymbol{y}} \sin\theta \sin\phi + \hat{\boldsymbol{z}} \cos\theta = \sqrt{\frac{4\pi}{3}} \sum_q (-1)^q \, Y_{1q}(\hat{\boldsymbol{n}}) \, \hat{\boldsymbol{e}}_{-q} \, . \tag{20}$$

Hence,

$$\hat{\boldsymbol{n}} Y_{\ell m}(\hat{\boldsymbol{n}}) = \sqrt{\frac{4\pi}{3}} \sum_q (-1)^q \, Y_{1q}(\hat{\boldsymbol{n}}) Y_{\ell m}(\hat{\boldsymbol{n}}) \, \hat{\boldsymbol{e}}_{-q} \, . \tag{21}$$

Using eq. (40) given in the Appendix, it follows that

$$Y_{1q}(\hat{\boldsymbol{n}}) Y_{\ell m}(\hat{\boldsymbol{n}}) = \sqrt{\frac{3(2\ell+1)}{4\pi}} \sum_{\ell'} \frac{1}{\sqrt{2\ell'+1}} \, \langle \ell \, m \, ; \, 1 \, q \, | \, \ell' \, , \, m+q \rangle \langle \ell \, 0 \, ; \, 1 \, 0 \, | \, \ell' \, 0 \rangle \, Y_{\ell' \, , \, m+q}(\hat{\boldsymbol{n}}) \, , \tag{22}$$

Only two terms, corresponding to $\ell' = \ell \pm 1$, can contribute to the sum over ℓ' since [cf. Table 2]:

$$\langle \ell \, 0 \, ; \, 1 \, 0 \, | \, \ell' \, 0 \rangle = \begin{cases} \left(\dfrac{\ell+1}{2\ell+1} \right)^{1/2} , & \text{for } \ell' = \ell + 1 \, , \\[2ex] 0 \, , & \text{for } \ell' \neq \ell \pm 1 \, , \\[2ex] -\left(\dfrac{\ell}{2\ell+1} \right)^{1/2} , & \text{for } \ell' = \ell - 1 \, . \end{cases} \tag{23}$$

[1] Henceforth, if left unspecified, sums over q will run over $q = -1, 0, +1$.

Inserting eq. (22) on the right hand side of eq. (21) and employing eq. (23) then yields

$$\hat{\boldsymbol{n}}Y_{\ell m}(\hat{\boldsymbol{n}}) = \sum_q (-1)^q\,\hat{\boldsymbol{e}}_{-q}\biggl\{\left(\frac{\ell+1}{2\ell+3}\right)^{1/2}\langle \ell\, m\,;\,1\,q\,|\,\ell+1\,,\,m+q\rangle\,Y_{\ell+1\,,\,m+q}(\hat{\boldsymbol{n}}) \\ -\left(\frac{\ell}{2\ell-1}\right)^{1/2}\langle \ell\, m\,;\,1\,q\,|\,\ell+1\,,\,m+q\rangle\,Y_{\ell-1\,,\,m+q}(\hat{\boldsymbol{n}})\biggr\}. \qquad (24)$$

It is convenient to rewrite eq. (24) with the help of the following two relations, which can be obtained from Table 2,

$$\langle \ell\, m\,;\,1\,q\,|\,\ell+1\,,\,m+q\rangle = -(-1)^q\left(\frac{2\ell+3}{2\ell+1}\right)^{1/2}\langle \ell+1\,,\,m+q\,;\,1\,,\,-q\,|\,\ell\, m\rangle\,, \qquad (25)$$

$$\langle \ell\, m\,;\,1\,q\,|\,\ell-1\,,\,m+q\rangle = -(-1)^q\left(\frac{2\ell-1}{2\ell+1}\right)^{1/2}\langle \ell-1\,,\,m+q\,;\,1\,,\,-q\,|\,\ell\, m\rangle\,. \qquad (26)$$

The end result is:

$$\hat{\boldsymbol{n}}Y_{\ell m}(\hat{\boldsymbol{n}}) = -\sum_q \hat{\boldsymbol{e}}_{-q}\biggl\{\left(\frac{\ell+1}{2\ell+1}\right)^{1/2}\langle \ell+1\,,\,m+q\,;\,1\,,\,-q\,|\,\ell\, m\rangle\,Y_{\ell+1\,,\,m+q}(\hat{\boldsymbol{n}}) \\ -\left(\frac{\ell}{2\ell+1}\right)^{1/2}\langle \ell-1\,,\,m+q\,;\,1\,,\,-q\,|\,\ell\, m\rangle\,Y_{\ell-1\,,\,m+q}(\hat{\boldsymbol{n}})\biggr\}. \qquad (27)$$

Using eq. (3) with $s=1$ and $\chi_{1\,q}=\hat{\boldsymbol{e}}_q$ and employing the notation of eq. (13), it follows that

$$\vec{\boldsymbol{Y}}_{\ell\,,\,\ell\pm 1\,,\,m}(\hat{\boldsymbol{n}}) = \sum_q \hat{\boldsymbol{e}}_{-q}\,\langle \ell\pm 1\,,\,m+q\,;\,1\,,\,-q\,|\,\ell\, m\rangle\,Y_{\ell\pm 1\,,\,m+q}(\hat{\boldsymbol{n}})\,, \qquad (28)$$

after relabeling the summation index by $q\to -q$. Hence, eq. (27) yields

$$\boxed{\hat{\boldsymbol{n}}Y_{\ell m}(\hat{\boldsymbol{n}}) = -\left(\frac{\ell+1}{2\ell+1}\right)^{1/2}\vec{\boldsymbol{Y}}_{\ell\,,\,\ell+1\,,\,m}(\hat{\boldsymbol{n}}) + \left(\frac{\ell}{2\ell+1}\right)^{1/2}\vec{\boldsymbol{Y}}_{\ell\,,\,\ell-1\,,\,m}(\hat{\boldsymbol{n}})} \qquad (29)$$

Finally, we examine $r\vec{\boldsymbol{\nabla}}Y_{\ell m}(\hat{\boldsymbol{n}})$. First, we introduce the gradient operator in a spherical basis, $\nabla_q = (\nabla_{+1}\,,\,\nabla_0\,,\,\nabla_{-1})$, where

$$\nabla_{+1} = \mp\frac{1}{\sqrt{2}}\left(\frac{\partial}{\partial x}\pm i\frac{\partial}{\partial y}\right) = \mp\frac{e^{\pm i\phi}}{\sqrt{2}}\left[\sin\theta\frac{\partial}{\partial r}+\frac{\cos\theta}{r}\frac{\partial}{\partial\theta}\pm\frac{i}{r\sin\theta}\frac{\partial}{\partial\phi}\right]\,, \qquad (30)$$

$$\nabla_0 = \frac{\partial}{\partial z} = \cos\theta\frac{\partial}{\partial r}-\frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\,. \qquad (31)$$

We can introduce a formal operator ∇_q on the Hilbert space by defining the coordinate space representation,

$$\langle\vec{\boldsymbol{x}}|\nabla_q|\ell\, m\rangle = \nabla_q Y_{\ell m}(\hat{\boldsymbol{n}})\,.$$

Note that ∇_q is a vector operator. We shall employ the Wigner-Eckart theorem (see, e.g., pp. 240–241 of Ref. [2]), which states that

$$\langle \ell'\, m' | \nabla_q | \ell\, m\rangle = \langle \ell\, m\,;\, 1\, q\,|\, \ell'\, m'\rangle\, \langle \ell' \| \nabla \| \ell\rangle\,, \tag{32}$$

where the reduced matrix element $\langle \ell \| \nabla \| \ell'\rangle$ is independent of q, m and m'. To evaluate the reduced matrix element, we consider the case of $q = m = m' = 0$. Then,

$$\langle \ell'\, 0 | \nabla_0 | \ell\, 0\rangle = \langle \ell\, 0\,;\, 1\, 0\,|\, \ell'\, 0\rangle\, \langle \ell' \| \nabla \| \ell\rangle\,.$$

Thus,

$$\langle \ell' \| \nabla \| \ell\rangle = \frac{\langle \ell'\, 0 | \nabla_0 | \ell\, 0\rangle}{\langle \ell\, 0\,;\, 1\, 0\,|\, \ell'\, 0\rangle}\,.$$

Inserting this result into eq. (32) yields

$$\langle \ell'\, m' | \nabla_q | \ell\, m\rangle = \frac{\langle \ell\, m\,;\, 1\, q\,|\, \ell'\, m'\rangle}{\langle \ell\, 0\,;\, 1\, 0\,|\, \ell'\, 0\rangle}\, \langle \ell'\, 0 | \nabla_0 | \ell\, 0\rangle\,. \tag{33}$$

We can evaluate $\langle \ell'\, 0 | \nabla_0 | \ell\, 0\rangle$ explicitly in the coordinate representation using eq. (31),

$$\langle \ell'\, 0 | \nabla_0 | \ell\, 0\rangle = -\frac{1}{r}\int d\Omega\, Y^*_{\ell'\,0}(\boldsymbol{\hat{n}})\sin\theta\,\frac{\partial}{\partial\theta}\, Y_{\ell\, 0}(\boldsymbol{\hat{n}})\,.$$

Using $Y_{\ell 0}(\boldsymbol{\hat{n}}) = [(2\ell+1)/(4\pi)]^{1/2}\, P_\ell(\cos\theta)$, and substituting $x \equiv \cos\theta$,

$$\langle \ell'\, 0 | \nabla_0 | \ell\, 0\rangle = \frac{\sqrt{(2\ell+1)(2\ell'+1)}}{2r}\int_{-1}^{1}(1-x^2)P_{\ell'}(x)P'_\ell(x)\, dx\,, \tag{34}$$

where $P'_\ell(x) = dP_\ell(x)/dx$. To evaluate eq. (34), we employ the recurrence relation,

$$(1-x^2)P'_\ell(x) = \ell P_{\ell-1}(x) - \ell x P_\ell(x)\,,$$

and the orthogonality relation of the Legendre polynomials,

$$\int_{-1}^{1} P_\ell(x)P_{\ell'}(x)\, dx = \frac{2}{2\ell+1}\,\delta_{\ell\ell'}\,.$$

It follows that

$$\langle \ell'\, 0 | \nabla_0 | \ell\, 0\rangle = \frac{\sqrt{(2\ell+1)(2\ell'+1)}}{2r}\left\{\frac{2\ell}{2\ell-1}\,\delta_{\ell',\ell-1} - \ell\int_{-1}^{1} xP_\ell(x)P_{\ell'}(x)\, dx\right\}. \tag{35}$$

To evaluate the remaining integral, we use $x = P_1(x)$ and the result of eq. (43) obtained in the Appendix to write:

$$\int_{-1}^{1} xP_\ell(x)P_{\ell'}(x)\, dx = \int_{-1}^{1} P_1(x)P_\ell(x)P_{\ell'}(x)\, dx = \frac{2}{2\ell'+1}\langle 1\, 0\,;\, \ell\, 0\,|\, \ell'\, 0\rangle^2\,.$$

Using eq. (23), the above integral is equal to

$$\int_{-1}^{1} xP_\ell(x)P_{\ell'}(x)\,dx = \frac{2(\ell+1)}{(2\ell+1)(2\ell+3)}\,\delta_{\ell',\ell+1} + \frac{2\ell}{(2\ell-1)(2\ell+1)}\,\delta_{\ell',\ell-1}\,.$$

Inserting this result back into eq. (35) yields

$$\begin{aligned}\langle \ell'\,0|\nabla_0|\ell\,0\rangle &= \frac{\sqrt{(2\ell+1)(2\ell'+1)}}{2r}\left\{\frac{2\ell(\ell+1)}{(2\ell-1)(2\ell+1)}\,\delta_{\ell',\ell-1} - \frac{2\ell(\ell+1)}{(2\ell+1)(2\ell+3)}\,\delta_{\ell',\ell+1}\right\}\\ &= \frac{\ell(\ell+1)}{r\sqrt{2\ell+1}}\left[\frac{1}{\sqrt{2\ell-1}}\,\delta_{\ell',\ell-1} - \frac{1}{\sqrt{2\ell+3}}\,\delta_{\ell',\ell+1}\right].\end{aligned}$$

Using eq. (33), it follows that:

$$\begin{aligned}\langle \ell'\,m'|\nabla_q|\ell\,m\rangle &= \frac{\langle \ell\,m\,;\,1\,q\,|\,\ell'\,m'\rangle}{\langle \ell\,0\,;\,1\,0\,|\,\ell'\,0\rangle}\,\frac{\ell(\ell+1)}{r\sqrt{2\ell+1}}\left[\frac{1}{\sqrt{2\ell-1}}\,\delta_{\ell',\ell-1} - \frac{1}{\sqrt{2\ell+3}}\,\delta_{\ell',\ell+1}\right]\\ &= -\frac{1}{r}\langle \ell\,m\,;\,1\,q\,|\,\ell'\,m'\rangle\left[(\ell+1)\sqrt{\frac{\ell}{2\ell-1}}\,\delta_{\ell',\ell-1} + \ell\sqrt{\frac{\ell+1}{2\ell+3}}\,\delta_{\ell',\ell+1}\right], \qquad (36)\end{aligned}$$

after using eq. (23) to evaluate $\langle \ell\,0\,;\,1\,0\,|\,\ell'\,0\rangle$.

We are now ready to evaluate $r\vec{\boldsymbol{\nabla}}Y_{\ell m}(\hat{\boldsymbol{n}})$. First, we insert a complete set of states to obtain

$$\begin{aligned}\nabla_q|\ell\,m\rangle &= \sum_{\ell',m'}|\ell'\,m'\rangle\langle\ell'\,m'|\nabla_q|\ell\,m\rangle\\ &= -\frac{1}{r}\sum_{\ell',m'}|\ell'\,m'\rangle\left\{\langle \ell\,m\,;\,1\,q\,|\,\ell'\,m'\rangle\left[(\ell+1)\sqrt{\frac{\ell}{2\ell-1}}\,\delta_{\ell',\ell-1} + \ell\sqrt{\frac{\ell+1}{2\ell+3}}\,\delta_{\ell',\ell+1}\right]\right\}.\end{aligned} \qquad (37)$$

Note that in the sum over m', only the terms corresponding to $m' = m+q$ survive, due to the presence of the Clebsch-Gordon coefficient $\langle \ell\,m\,;\,1\,q\,|\,\ell'\,m'\rangle$. Likewise, in the sum over ℓ', only the terms corresponding to $\ell' = \ell\pm 1$ survive. In the coordinate representation, eq. (37) is equivalent to

$$\nabla_q Y_{\ell m}(\hat{\boldsymbol{n}}) = -\frac{1}{r}\sum_{\ell'} Y_{\ell',m+q}(\hat{\boldsymbol{n}})\left\{\langle \ell\,m\,;\,1\,q\,|\,\ell'\,,\,m{+}q\rangle\left[(\ell+1)\sqrt{\frac{\ell}{2\ell-1}}\,\delta_{\ell',\ell-1} + \ell\,\sqrt{\frac{\ell+1}{2\ell+3}}\,\delta_{\ell',\ell+1}\right]\right\}.$$

In analogy with eq. (18), we have

$$\vec{\boldsymbol{\nabla}} = \sum_q (-1)^q\,\hat{\boldsymbol{e}}_{-q}\nabla_q\,.$$

Hence, it follows that

$$-r\vec{\boldsymbol{\nabla}}Y_{\ell m}(\hat{\boldsymbol{n}}) = \sum_q (-1)^q\,\hat{\boldsymbol{e}}_{-q}\Biggl\{(\ell+1)\sqrt{\frac{\ell}{2\ell-1}}\,\langle \ell\, m\,;\,1\,q\,|\,\ell-1\,,\,m+q\rangle Y_{\ell-1,m+q}(\hat{\boldsymbol{n}})$$
$$+\ell\sqrt{\frac{\ell+1}{2\ell+3}}\,\langle \ell\, m\,;\,1\,q\,|\,\ell+1\,,\,m+q\rangle Y_{\ell+1,m+q}(\hat{\boldsymbol{n}})\Biggr\}\,.$$

It is convenient to employ eqs. (25) and (26) and rewrite the above result as

$$r\vec{\boldsymbol{\nabla}}Y_{\ell m}(\hat{\boldsymbol{n}}) = \sum_q \hat{\boldsymbol{e}}_{-q}\Biggl\{(\ell+1)\sqrt{\frac{\ell}{2\ell+1}}\,\langle \ell-1\,,\,m+q\,;\,1\,,\,-q\,|\,\ell\, m\rangle Y_{\ell-1,m+q}(\hat{\boldsymbol{n}})$$
$$+\ell\sqrt{\frac{\ell+1}{2\ell+1}}\,\langle \ell+1\,,\,m+q\,;\,1\,,\,-q\,|\,\ell\, m\rangle Y_{\ell+1,m+q}(\hat{\boldsymbol{n}})\Biggr\}\,.$$

Finally, using eq. (28), we end up with

$$\boxed{r\vec{\boldsymbol{\nabla}}Y_{\ell m}(\hat{\boldsymbol{n}}) = (\ell+1)\sqrt{\frac{\ell}{2\ell+1}}\,\vec{\boldsymbol{Y}}_{\ell,\ell-1,m}(\hat{\boldsymbol{n}}) + \ell\sqrt{\frac{\ell+1}{2\ell+1}}\,\vec{\boldsymbol{Y}}_{\ell,\ell+1,m}(\hat{\boldsymbol{n}})} \tag{38}$$

which is known in the literature as the *gradient formula*.

We can now use eqs. (29) and (38) to solve for $\vec{\boldsymbol{Y}}_{\ell,\ell+1,m}(\hat{\boldsymbol{n}})$ and $\vec{\boldsymbol{Y}}_{\ell,\ell-1,m}(\hat{\boldsymbol{n}})$ in terms of $\hat{\boldsymbol{n}}Y_{\ell m}(\hat{\boldsymbol{n}})$ and $r\vec{\boldsymbol{\nabla}}Y_{\ell m}(\hat{\boldsymbol{n}})$. Since these are linear equations, they are easily inverted, and we find

$$\vec{\boldsymbol{Y}}_{\ell,\ell+1,m}(\hat{\boldsymbol{n}}) = \frac{1}{\sqrt{(\ell+1)(2\ell+1)}}\left[-(\ell+1)\hat{\boldsymbol{n}} + r\vec{\boldsymbol{\nabla}}\right]Y_{\ell m}(\hat{\boldsymbol{n}})\,, \qquad \text{for } \ell = 0,1,2,3,\ldots,$$

$$\vec{\boldsymbol{Y}}_{\ell,\ell-1,m}(\hat{\boldsymbol{n}}) = \frac{1}{\sqrt{\ell(2\ell+1)}}\left[\ell\hat{\boldsymbol{n}} + r\vec{\boldsymbol{\nabla}}\right]Y_{\ell m}(\hat{\boldsymbol{n}})\,, \qquad \text{for } \ell = 1,2,3,\ldots,$$

which are equivalent to the results of eqs. (9) and (10) previously obtained. In addition, we also have eq. (19), which we can rewrite as

$$\vec{\boldsymbol{Y}}_{\ell,\ell,m} = \frac{-ir}{\sqrt{\ell(\ell+1)}}\,\hat{\boldsymbol{n}}\times\vec{\boldsymbol{\nabla}}Y_{\ell m}(\hat{\boldsymbol{n}})\,, \qquad \text{for } \ell = 1,2,3,\ldots,.$$

Thus, we have identified the three linearly independent vector spherical harmonics in terms of differential vector operators acting on $Y_{\ell m}(\hat{\boldsymbol{n}})$. For the special case of $\ell = 0$, only one vector spherical harmonic, $\vec{\boldsymbol{Y}}_{010}(\hat{\boldsymbol{n}}) = (-\hat{\boldsymbol{n}} + r\vec{\boldsymbol{\nabla}})Y_{\ell m}(\hat{\boldsymbol{n}})$, survives.

In books, one often encounters the vector spherical harmonic defined by $\hat{\boldsymbol{n}}\times\vec{\boldsymbol{L}}\,Y_{\ell m}(\hat{\boldsymbol{n}})$. However, this is not independent of the vector spherical harmonics obtained above, since

$$\hat{\boldsymbol{n}}\times\vec{\boldsymbol{L}}\,Y_{\ell m}(\hat{\boldsymbol{n}}) = -i\hbar r\,\hat{\boldsymbol{n}}\times(\hat{\boldsymbol{n}}\times\vec{\boldsymbol{\nabla}})Y_{\ell m}(\hat{\boldsymbol{n}}) = -i\hbar r\left[\hat{\boldsymbol{n}}\frac{\partial}{\partial r} - \vec{\boldsymbol{\nabla}}\right]Y_{\ell m}(\hat{\boldsymbol{n}}) = i\hbar r\,\vec{\boldsymbol{\nabla}}Y_{\ell m}(\hat{\boldsymbol{n}})\,.$$

An alternative method for deriving the gradient formula [obtained in eq. (38)] is to evaluate $\hat{\boldsymbol{n}} \times \vec{\boldsymbol{L}}\, Y_{\ell m}(\hat{\boldsymbol{n}})$ using the same technique employed in the computation of $\hat{\boldsymbol{n}} Y_{\ell m}(\hat{\boldsymbol{n}})$ given in this section. However, this calculation is much more involved and involves a product of four Clebsch-Gordon coefficients. A certain sum involving a product of three Clebsch-Gordon coefficients needs to be performed in closed form. This summation can be done (e.g., see Ref. [11] for the gory details), but the computation is much more involved than the simple analysis presented in this section based on the Wigner-Eckart theorem.

Appendix: An integral of a product of three spherical harmonics

In this Appendix, we state a number of results that are derived on pp. 216–217 of Ref. [2]. The product of two spherical harmonics can be obtained via an important expansion known as the Clebsch-Gordon series,

$$Y_{\ell_1 m_1}(\hat{\boldsymbol{n}})\, Y_{\ell_2 m_2}(\hat{\boldsymbol{n}}) = \sum_{\ell, m} \sqrt{\frac{(2\ell_1+1)(2\ell_2+1)}{4\pi(2\ell+1)}} \langle \ell_1\, m_1\,;\, \ell_2\, m_2 \,|\, \ell\, m\rangle \langle \ell_1\, 0\,;\, \ell_2\, 0 \,|\, \ell\, 0\rangle\, Y_{\ell m}(\hat{\boldsymbol{n}}). \tag{39}$$

The sum over m can be performed using eq. (2). Only one term survives (corresponding to $m = m_1 + m_2$),

$$Y_{\ell_1 m_1}(\hat{\boldsymbol{n}})\, Y_{\ell_2 m_2}(\hat{\boldsymbol{n}}) = \sum_{\ell} \sqrt{\frac{(2\ell_1+1)(2\ell_2+1)}{4\pi(2\ell+1)}} \langle \ell_1\, m_1\,;\, \ell_2\, m_2 \,|\, \ell\ m_1{+}m_2\rangle \langle \ell_1\, 0\,;\, \ell_2\, 0 \,|\, \ell\, 0\rangle\, Y_{\ell,\, m_1+m_2}(\hat{\boldsymbol{n}}). \tag{40}$$

Note that $\langle \ell_1\, m_1\,;\, \ell_2\, m_2 \,|\, \ell\ m_1 + m_2\rangle = 0$ unless the two conditions, $|\ell_1 - \ell_2| \leq \ell \leq \ell_1 + \ell_2$ and $|m_1 + m_2| \leq \ell$ are both satisfied. This is simply a consequence of the rules for the addition of angular momentum in quantum mechanics. Consequently, the sum over ℓ in eq. (40) can be taken over the range of integer values that satisfy

$$\max\bigl\{|\ell_1 - \ell_2|\,,\, m_1 + m_2\bigr\} \leq \ell \leq \ell_1 + \ell_2\,.$$

If we multiply eq. (39) by $Y^*_{\ell_3 m_3}(\hat{\boldsymbol{n}})$ and then integrate over the solid angle using the orthogonality of the spherical harmonics,

$$\int Y_{\ell m}(\theta, \phi)\, Y^*_{\ell' m'}(\theta, \phi)\, d\Omega = \delta_{\ell\ell'}\, \delta_{mm'}\,, \tag{41}$$

one easily obtains the integral of a product of three spherical harmonics,

$$\int Y_{\ell_1 m_1}(\theta, \phi)\, Y_{\ell_2 m_2}(\theta, \phi)\, Y^*_{\ell_3 m_3}(\theta, \phi)\, d\Omega = \sqrt{\frac{(2\ell_1+1)(2\ell_2+1)}{4\pi(2\ell_3+1)}}\, \langle \ell_1\, m_1\,;\, \ell_2\, m_2 \,|\, \ell_3\, m_3\rangle\, \langle \ell_1\, 0\,;\, \ell_2\, 0 \,|\, \ell_3\, 0\rangle\,. \tag{42}$$

Using $Y_{\ell 0}(\hat{\boldsymbol{n}}) = \left[(2\ell+1)/(4\pi)\right]^{1/2} P_\ell(\cos\theta)$, the following special case of eq. (42) is then obtained,

$$\int_{-1}^{1} P_{\ell_1}(x) P_{\ell_2}(x) P_{\ell_3}(x)\, dx = \frac{2}{2\ell_3+1}\, \langle \ell_1\, 0\,;\, \ell_2\, 0 \,|\, \ell_3\, 0\rangle^2\,. \tag{43}$$

Bibliography

The basics of angular momentum theory, Clebsch-Gordon coefficients, spherical tensors and the Wigner-Eckart theorem are treated in most quantum mechanics textbooks. Pedagogical treatments can be found in the following two well-known textbooks:

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