Maxwell Equations (cgs units)

$$\vec{\nabla}\cdot\vec{E} = 4\pi\rho \qquad \vec{\nabla}\cdot\vec{B} = 0$$

$$\vec{\nabla}\times\vec{E} = -\frac{1}{c}\frac{\partial\vec{B}}{\partial t} \qquad \vec{\nabla}\times\vec{B} = \frac{4\pi}{c}\vec{J} + \frac{1}{c}\frac{\partial\vec{E}}{\partial t}$$

$$\vec{\nabla}\cdot\frac{\partial\vec{E}}{\partial t} = 4\pi\frac{\partial\rho}{\partial t}$$

$$= \vec{\nabla}\cdot\left[c(\vec{\nabla}\times\vec{B}) - 4\pi\vec{J}\right]$$

$$= -4\pi\vec{\nabla}\cdot\vec{J}$$

$\Longrightarrow$

$$\vec{\nabla}\cdot\vec{J} + \frac{\partial\rho}{\partial t} = 0$$

continuity equation

$$Q = \int_V \rho\, d^3x$$

$$-\frac{dQ}{dt} = \oint_S \vec{J}\cdot d\vec{a}$$

rate of change loss in volume V =

rate of flow of change (current) across S.

Gauge invariance of Maxwell's eqs.

$$\vec{\nabla}\cdot\vec{B}=0 \quad \Longrightarrow \quad \boxed{\vec{B}=\vec{\nabla}\times\vec{A}}$$

$$\vec{\nabla}\times\left(\vec{E}+\frac{1}{c}\frac{\partial\vec{A}}{\partial t}\right)=0$$

$$\Longrightarrow \quad \vec{E}+\frac{1}{c}\frac{\partial\vec{A}}{\partial t}=-\vec{\nabla}\Phi$$

$$\boxed{\vec{E}=-\vec{\nabla}\Phi-\frac{1}{c}\frac{\partial\vec{A}}{\partial t}}$$

Φ, $\vec{A}$ are not unique

The $\vec{E}$, $\vec{B}$ are unchanged by gauge transformations

$$\vec{A}\longrightarrow\vec{A}+\vec{\nabla}\Lambda(\vec{x},t)$$

$$\Phi\longrightarrow\Phi-\frac{1}{c}\frac{\partial\Lambda(\vec{x},t)}{\partial t}$$

$\Lambda(\vec{x},t)$ is a smooth function of $\vec{x}$, t.

Rewrite Maxwell's eqs. in terms of $\vec{A}$, Φ

$$\vec{\nabla}\times(\vec{\nabla}\times\vec{A}) = \vec{\nabla}(\vec{\nabla}\cdot\vec{A}) - \vec{\nabla}^2\vec{A}$$

where

$$\vec{\nabla}^2\vec{A} \equiv (\vec{\nabla}^2 A_x, \vec{\nabla}^2 A_y, \vec{\nabla}^2 A_z)$$

$$\vec{\nabla}^2\vec{A} - \frac{1}{c^2}\frac{\partial^2\vec{A}}{\partial t^2} - \vec{\nabla}\left(\vec{\nabla}\cdot\vec{A} + \frac{1}{c}\frac{\partial\Phi}{\partial t}\right) = \frac{-4\pi\vec{J}}{c}$$

d'Alembertian:

$$\Box \equiv \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \vec{\nabla}^2$$

$$\boxed{\Box\vec{A} + \vec{\nabla}\left(\vec{\nabla}\cdot\vec{A} + \frac{1}{c}\frac{\partial\Phi}{\partial t}\right) = \frac{4\pi\vec{J}}{c}}$$

$$\vec{\nabla}\cdot\vec{E} = 4\pi\rho \implies$$

$$\boxed{\vec{\nabla}\cdot\left(\vec{\nabla}\Phi + \frac{1}{c}\frac{\partial\vec{A}}{\partial t}\right) = -4\pi\rho}$$

We can choose a gauge

1. Lorenz gauge $\quad \vec{\nabla}\cdot\vec{A} + \frac{1}{c}\frac{\partial\Phi}{\partial t} = 0$
2. Coulomb gauge $\quad \vec{\nabla}\cdot\vec{A} = 0$

For example, let

$$\vec{A}' = \vec{A} + \vec{\nabla}\Lambda \,, \quad \Phi' = \Phi - \frac{1}{c}\frac{\partial \Lambda}{\partial t}$$

Then, $\vec{\nabla}\cdot\vec{A}' + \frac{1}{c}\frac{\partial \Phi'}{\partial t} = 0 \implies$

$$\Box\Lambda = \vec{\nabla}\cdot\vec{A} + \frac{1}{c}\frac{\partial \Phi}{\partial t} \quad \text{inhomogeneous wave equation}$$

(aside: Λ is not uniquely fixed. If you have established the Lorenz gauge, then further gauge transformations with $\Box\Lambda = 0$ leave you in the Lorenz gauge)

If $\vec{A}', \Phi'$ satisfy $\vec{\nabla}\cdot\vec{A}' = 0$

then $\vec{\nabla}\cdot\vec{A} + \vec{\nabla}^2\Lambda = 0$

$\implies \vec{\nabla}^2\Lambda = -\vec{\nabla}\cdot\vec{A}$ satisfies Poisson's eq.

Solution:

$$\Lambda(\vec{x},t) = \frac{1}{4\pi}\int d^3x' \, \frac{\vec{\nabla}'\cdot\vec{A}(\vec{x}',t)}{|\vec{x}-\vec{x}'|}$$

(aside: if $\vec{\nabla}^2\Lambda = 0$, then you stay in Coulomb gauge $\implies \Lambda = \Lambda(t)$ independent of $\vec{x}$)

(i) Lorenz gauge

$$\Box \vec{A} = \frac{4\pi}{c} \vec{J}$$

$$\Box \Phi = 4\pi \rho$$

inhomogeneous wave equations.

Note: if $\vec{J} = \rho = 0$, then

$$\Box \vec{A} = 0$$
$$\Box \Phi = 0$$
$\Longrightarrow$ electromagnetic waves

$$\Box \vec{E} = 0$$
$$\Box \vec{B} = 0$$

(ii) Coulomb gauge

detour:

Any vector $\vec{V}$ can be split up into a "transverse" and "longitudinal" piece.

$$\vec{V} = \vec{V}_{long} + \vec{V}_{tr} \quad \text{such that}$$

$\vec{\nabla} \cdot \vec{V}_{tr} = 0$ solenoidal

$\vec{\nabla} \times \vec{V}_{long} = 0$ irrotational origin of names

$$\vec{V}(\vec{x},t) = \vec{\varepsilon}\, e^{i(\vec{k}\cdot\vec{x}-\omega t)}$$

$$\Rightarrow \quad \vec{k}\cdot\vec{V}_{tr} = 0$$

$$\vec{k}\times\vec{V}_{long} = 0$$

$$\vec{V}_{long} = \vec{\nabla}\,\vec{\nabla}^{-2}(\vec{\nabla}\cdot\vec{V})$$

inverse Laplacian

$$\vec{V}_{long} = \vec{\nabla}\psi \quad \text{where} \quad \psi = \vec{\nabla}^{-2}(\vec{\nabla}\cdot\vec{V})$$

$$\Rightarrow \vec{\nabla}\times\vec{V}_{long} = 0$$

$$\vec{V}_{tr} = \vec{V} - \vec{V}_{long}$$

$$\vec{\nabla}\cdot\vec{V}_{tr} = \vec{\nabla}\cdot\vec{V} - \vec{\nabla}\cdot\vec{V}_{long}$$

$$= \vec{\nabla}\cdot\vec{V} - \vec{\nabla}^2\vec{\nabla}^{-2}(\vec{\nabla}\cdot\vec{V})$$

$$= \vec{\nabla}\cdot\vec{V} - \vec{\nabla}\cdot\vec{V}$$

$$= 0$$

as long as I can show that $\vec{\nabla}^2\vec{\nabla}^{-2} = \mathbb{1}$.

Definition:

$$\vec{\nabla}^{-2} f(\vec{x}) = -\frac{1}{4\pi}\int d^3x' \, \frac{f(\vec{x}')}{|\vec{x}-\vec{x}'|}$$

non local (depends on all $\vec{x}'$)

$$\vec{\nabla}^2 \vec{\nabla}^{-2} f(x) = -\frac{1}{4\pi}\int d^3x' f(\vec{x}') \, \vec{\nabla}^2\left(\frac{1}{|\vec{x}-\vec{x}'|}\right)$$

recall: $\vec{\nabla}^2\left(\frac{1}{|\vec{x}-\vec{x}'|}\right) = -4\pi\,\delta^3(\vec{x}-\vec{x}')$

$$\vec{\nabla}^2 \vec{\nabla}^{-2} f(\vec{x}) = \int d^3x' f(\vec{x}')\,\delta^3(\vec{x}-\vec{x}')$$

$$= f(\vec{x})$$

$$\vec{\nabla}^2 \vec{\nabla}^{-2} = 1$$

exercise: show that $\vec{\nabla}^{-2}\vec{\nabla}^2 = 1$.

Conclusion

$$\boxed{\vec{V}_{long} = -\frac{1}{4\pi}\vec{\nabla}\int d^3x' \, \frac{\vec{\nabla}' \cdot \vec{V}(\vec{x}')}{|\vec{x}-\vec{x}'|}}$$

exercise: show that

$$\boxed{\begin{aligned}\vec{V}_{tr} &= -\vec{\nabla}\times\vec{\nabla}^{-2}(\vec{\nabla}\times\vec{V}) \\ &= \frac{1}{4\pi}\vec{\nabla}\times\int\frac{\vec{\nabla}'\times\vec{V}(\vec{x}')}{|\vec{x}-\vec{x}'|}\,d^3x'\end{aligned}}$$

proof: show that $\vec{V} = \vec{V}_{long} + \vec{V}_{tr}$.

Back to Coulomb gauge $(\vec{\nabla}\cdot\vec{A}=0)$

$$\boxed{\begin{aligned}&\Box\vec{A} + \frac{1}{c}\vec{\nabla}\left(\frac{\partial\Phi}{\partial t}\right) = \frac{4\pi}{c}\vec{J} \\ &\vec{\nabla}^2\Phi = -4\pi\rho\end{aligned}}$$

Poisson's equation for Φ.

$$\begin{aligned}\Phi(\vec{x},t) &= -4\pi\vec{\nabla}^{-2}\rho(\vec{x},t) \\ &= \int d^3x'\,\frac{\rho(\vec{x}',t)}{|\vec{x}-\vec{x}'|}\end{aligned}$$

instantaneous Coulomb potential.

$$\vec{\nabla}\frac{\partial\Phi}{\partial t} = \vec{\nabla}\int d^3x' \, \frac{\partial\rho/\partial t}{|\vec{x}-\vec{x}'|}$$

$$= -\vec{\nabla}\int d^3x' \, \frac{\vec{\nabla}'\cdot\vec{J}(\vec{x}',t)}{|\vec{x}-\vec{x}'|}$$

using continuity equation.

$$= 4\pi \vec{J}_{long}$$

$$\Longrightarrow$$

$$\Box\vec{A} = \frac{4\pi}{c}(\vec{J}-\vec{J}_{long}) = \frac{4\pi}{c}\vec{J}_{tr}$$

$$\boxed{\Box\vec{A} = \frac{4\pi}{c}\vec{J}_{tr}}$$

Comparison of gauges

	Coulomb	Lorenz
gauge condition	$\vec{\nabla}\cdot\vec{A}=0$	$\vec{\nabla}\cdot\vec{A}+\frac{1}{c}\frac{\partial\Phi}{\partial t}=0$
Φ	$\vec{\nabla}^2\Phi=-4\pi\rho$	$\Box\Phi=4\pi\rho$
$\vec{A}$	$\Box\vec{A}=\frac{4\pi}{c}\vec{J}_{tr}$	$\Box\vec{A}=\frac{4\pi}{c}\vec{J}$

The free electromagnetic field $\quad (\rho = \vec{J} = 0)$

$$\vec{\nabla}\cdot\vec{E} = 0 \qquad \vec{\nabla}\times\vec{E} = -\frac{1}{c}\frac{\partial\vec{B}}{\partial t}$$

$$\vec{\nabla}\cdot\vec{B} = 0 \qquad \vec{\nabla}\times\vec{B} = \frac{1}{c}\frac{\partial\vec{E}}{\partial t}$$

$$\vec{B} = \vec{\nabla}\times\vec{A}$$

$$\vec{E} = -\vec{\nabla}\Phi - \frac{1}{c}\frac{\partial\vec{A}}{\partial t}$$

In the Coulomb gauge,

$$\Phi(\vec{x},t) = \int d^3x' \, \frac{\rho(\vec{x}',t)}{|\vec{x}-\vec{x}'|} = 0$$

$$\Box\vec{A} = \frac{4\pi}{c}\vec{J}_{tr} = 0$$

$$\Rightarrow \quad \vec{B} = \vec{\nabla}\times\vec{A}$$

$$\vec{E} = -\frac{1}{c}\frac{\partial\vec{A}}{\partial t}$$

$$\vec{A}(\vec{x},t) = \frac{1}{(2\pi)^3}\int d^3k \, \vec{A}_0(\vec{k},t)\, e^{i\vec{k}\cdot\vec{x}}$$

$$\Box\vec{A}(\vec{x},t) = 0 \qquad \Box = \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \vec{\nabla}^2$$

$$\frac{1}{c^2}\frac{d^2}{dt^2}\vec{A}_0(\vec{k},t) = -k^2\vec{A}_0(\vec{k},t)$$

$$\boxed{\omega = ck}$$

dispersion relation

solutions:

$$\vec{A}_0(\vec{k},t) \sim e^{\pm i\omega t}$$

$$\vec{A}(\vec{x},t) = \frac{1}{(2\pi)^3}\int d^3k\,\left[\vec{a}(\vec{k})e^{-i\omega t} + \vec{b}(\vec{k})e^{i\omega t}\right]e^{i\vec{k}\cdot\vec{x}}$$

But $\vec{A}(\vec{x},t)$ is a real vector field $(\vec{A}^* = \vec{A})$

$$\vec{b}(-\vec{k}) = \vec{a}^*(\vec{k})$$

(after taking $\vec{k} \to -\vec{k}$ in the integral of $\vec{A}^*(\vec{x},t)$)

$$\boxed{\vec{A}(\vec{x},t) = \frac{1}{(2\pi)^3}\int d^3k\,\left[\vec{a}(\vec{k})e^{i(\vec{k}\cdot\vec{x}-\omega t)} + \vec{a}^*(\vec{k})e^{-i(\vec{k}\cdot\vec{x}-\omega t)}\right]}$$

$\Longrightarrow$

$$\vec{E}(\vec{x},t) = \frac{1}{(2\pi)^3}\int d^3k\left[\vec{E}_0(\vec{k})e^{i(\vec{k}\cdot\vec{x}-\omega t)} + \text{c.c.}\right]$$

c.c. = complex conjugate

where $\vec{E}_0(\vec{k}) = \frac{i\omega}{c}\vec{a}(\vec{k}) = ik\,\omega(\vec{k})$, $k = |\vec{k}|$

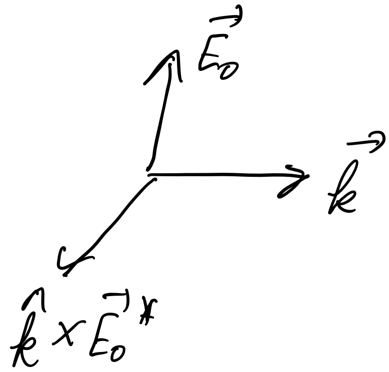
$$\vec{B}(\vec{x},t) = \frac{1}{(2\pi)^3}\int d^3k\left[\vec{B}_0(\vec{k})e^{i(\vec{k}\cdot\vec{x}-\omega t)} + \text{c.c.}\right]$$

$(\hat{k} = \vec{k}/k)$ $\qquad \vec{B}_0(\vec{k}) = i\vec{k}\times\vec{a}(\vec{k}) = \hat{k}\times\vec{E}_0(\vec{k})$

We still have to impose $\vec{\nabla}\cdot\vec{E} = \vec{\nabla}\cdot\vec{B} = 0$

$$\vec{\nabla}\cdot\vec{E} = 0 \implies \vec{k}\cdot\vec{E}_0 = 0$$

$\vec{\nabla}\cdot\vec{B} = 0$ automatic since $\vec{k}\cdot\vec{B}_0 = \vec{k}\cdot(\hat{k}\times\vec{E}_0) = 0$.



Electromagnetic waves are transverse.

For any mode $\vec{k}$

$(\hat{\varepsilon}_1, \hat{\varepsilon}_2, \hat{k})$ mutually orthogonal unit vectors in a complex vector space

$$\hat{\varepsilon}_1\cdot\hat{\varepsilon}_1^* = \hat{\varepsilon}_2\cdot\hat{\varepsilon}_2^* = \hat{k}\cdot\hat{k} = 1$$

$$\hat{\varepsilon}_\lambda\cdot\hat{\varepsilon}_{\lambda'}^* = \delta_{\lambda\lambda'} \qquad \lambda, \lambda' = 1, 2$$

$$\hat{\varepsilon}_\lambda\cdot\hat{k} = \hat{\varepsilon}_\lambda^*\cdot\hat{k} = 0 \qquad \lambda = 1, 2$$

$$\vec{E}_0(\vec{k}) = \sum_{\lambda=1,2} E_0(\vec{k}, \lambda)\, \hat{\varepsilon}_{\vec{k},\lambda}$$

$\hat{\varepsilon}_\lambda$ polarization vectors.

$$\vec{E}(\vec{x},t) = \sum_\lambda \int \frac{d^3k}{(2\pi)^3} \left[E_0(\vec{k},\lambda)\hat{\varepsilon}_\lambda(\vec{k}) e^{i(\vec{k}\cdot\vec{x}-\omega t)} + c.c. \right]$$

λ runs over two polarization states

$\vec{E}$ is a real field

$$\hat{\varepsilon}_\lambda(-\vec{k}) = \hat{\varepsilon}_\lambda^*(\vec{k})$$

$$\vec{\nabla}\cdot\vec{E} = 0 \implies \vec{k}\cdot\hat{\varepsilon}_\lambda(\vec{k}) = 0$$

Examples

(i) linear polarization

e.g. $\hat{k} = (0,0,1) = \hat{\varepsilon}_0$ not a polarization vector

$\hat{\varepsilon}_x = (1,0,0)$

$\hat{\varepsilon}_y = (0,1,0)$

$\vec{k} = \hat{k}|\vec{k}|$ wave number of mode

(ii) circular polarization

e.g. $\hat{k} = (0,0,1) = \hat{\varepsilon}_0$

$$\hat{\varepsilon}_\pm = \mp\frac{1}{\sqrt{2}}(\hat{\varepsilon}_x \pm i\hat{\varepsilon}_y)$$

$$\hat{\varepsilon}_m^*(k)\cdot\hat{\varepsilon}_{m'}(k) = \delta_{mm'}$$

$$\hat{\varepsilon}_m^*(k) = (-1)^m\hat{\varepsilon}_{-m}(k)$$

The $\pm$ convention is consistent with phase convention of $Y_{1m}(\theta,\phi)$

$\hat{\varepsilon}_+$ positive helicity (left-circular polarized)

$\hat{\varepsilon}_-$ negative helicity (right-circular polarized)

"optics convention"

Complex field

$$\vec{E}(\vec{x},t) = \mp E_0 (\hat{\varepsilon}_x \pm i \hat{\varepsilon}_y) e^{i(\vec{k}\cdot\vec{x}-\omega t)}, \quad \vec{k} = k\hat{z}$$

(E_0 real)

physical field: $\mathrm{Re}\,\vec{E}$

$$\mathrm{Re}\,E_x(\vec{x},t) = \mp E_0 \cos(kz-\omega t)$$
$$\mathrm{Re}\,E_y(\vec{x},t) = E_0 \sin(kz-\omega t)$$

$$E_0^2 = [\mathrm{Re}\,E_x]^2 + [\mathrm{Re}\,E_y]^2$$

$\Rightarrow$ $\mathrm{Re}\,\vec{E}$ traces out a circle

upper sign: clockwise rotation when viewing an approaching wave (right)

lower sign: anticlockwise rotation (left)

optics convention

(iii) elliptic polarization

$$\vec{E}_0(\vec{k}) = \sum_{\lambda} E_0(\vec{k},\lambda)\,\hat{\varepsilon}_\lambda(\vec{k})$$
$$= \hat{\varepsilon}_x a_1 e^{i\delta_1} + \hat{\varepsilon}_y a_2 e^{i\delta_2}$$

(where a_1, a_2 are real and non-negative)
δ_1, δ_2 are real

$$\vec{E} = \mathrm{Re}\left(\vec{E}_0 e^{i(kz-\omega t)}\right) \quad \text{where } \phi = \omega t - kz$$

$$E_x = a_1 \cos(\phi - \delta_1)$$
$$E_y = a_2 \cos(\phi - \delta_2)$$

$$\cos^{-1} A - \cos^{-1} B = \mp \cos^{-1}\left(AB + \sqrt{1-A^2}\sqrt{1-B^2}\right)$$

$$\begin{array}{ll} - & A \geq B \\ + & A < B \end{array}$$

<u>exercise:</u>

$$\frac{E_x^2}{a_1^2} + \frac{E_y^2}{a_2^2} - \frac{2E_x E_y}{a_1 a_2}\cos\delta = \sin^2\delta$$

$$\delta \equiv \delta_2 - \delta_1$$

$\vec{E}$ traces out an ellipse.

$\{a_1, a_2, \delta\}$ are related to the Stokes parameters (S_i)

$$S_0 = a_1^2 + a_2^2 \qquad S_2 = 2a_1 a_2 \cos\delta$$
$$S_1 = a_1^2 - a_2^2 \qquad S_3 = 2a_1 a_2 \sin\delta$$

note: $S_0^2 = S_1^2 + S_2^2 + S_3^2$.

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complex fields

$$\vec{E}(\vec{x},t) = \sum_\lambda \int \frac{d^3k}{(2\pi)^3}\, E_0(\vec{k},\lambda)\,\hat{\mathcal{E}}_\lambda(\vec{k})\, e^{i(\vec{k}\cdot\vec{x} - \omega t)}$$

energy density averaged over a cycle (cgs units)

$$\langle u \rangle = \frac{1}{16\pi}\left[\vec{E}\cdot\vec{E}^* + \vec{B}\cdot\vec{B}^*\right]$$ in vacuum

Poynting vector

$$\langle \vec{S}_c \rangle = \frac{c}{8\pi} \vec{E} \times \vec{B}^* \quad , \quad \langle \vec{S} \rangle = Re \langle \vec{S}_c \rangle$$

complex notation

$$\vec{A}(t) = \vec{A}_0 e^{-i\omega t}$$

$$\vec{B}(t) = \vec{B}_0 e^{-i\omega t}$$

$$Re\, \vec{A}(t) \cdot Re\, \vec{B}(t) =$$

$$\frac{1}{4}\left(\vec{A}_0 e^{-i\omega t} + \vec{A}_0^* e^{i\omega t}\right) \cdot \left(\vec{B}_0 e^{-i\omega t} + \vec{B}_0 e^{i\omega t}\right)$$

$$= \frac{1}{4}\left(\vec{A}_0 \cdot \vec{B}_0^* + \vec{A}_0^* \cdot \vec{B}_0\right) + \frac{1}{4}\left(\vec{A}_0 \cdot \vec{B}_0 e^{-2i\omega t} + c.c.\right)$$

$$\langle Re\, \vec{A}(t) \cdot Re\, \vec{B}(t) \rangle = \frac{1}{2} Re\left(\vec{A} \cdot \vec{B}^*\right)$$

since $\langle e^{-2i\omega t} \rangle = \langle e^{2i\omega t} \rangle = 0$.

special case: $\vec{A} = \vec{B}$

$$\langle (Re\, \vec{A})^2 \rangle = \frac{1}{2} \vec{A} \cdot \vec{A}^*$$

Monochromatic EM waves (Coulomb gauge)

$$\vec{A}(\vec{x},t) = \vec{A}_0 e^{i(\vec{k}\cdot\vec{x}-\omega t)}$$

(take real part to get the physical field)

$$\Box \vec{A} = 0 \qquad \text{wave equation when } \rho = \vec{J} = 0$$

$$\Longrightarrow \quad \left(-\frac{\omega^2}{c^2} + k^2\right)\vec{A}_0 e^{i(\vec{k}\cdot\vec{x}-\omega t)} = 0$$

$$\Longrightarrow \quad \omega = ck \qquad \text{(dispersion relation)}$$

$\omega = 2\pi\nu$ $\quad$ ν = frequency

$k = \dfrac{2\pi}{\lambda}$ $\quad$ λ = wavelength

$$\vec{E} = -\frac{1}{c}\frac{\partial \vec{A}}{\partial t}$$

$$= \frac{i\omega}{c}\vec{A}_0 e^{i(\vec{k}\cdot\vec{x}-\omega t)}$$

$$= \vec{E}_0 e^{i(\vec{k}\cdot\vec{x}-\omega t)} \qquad \vec{E}_0 = \frac{i\omega}{c}\vec{A}_0$$

$$\vec{B} = \hat{k}\times\vec{E} = \vec{B}_0 e^{i(\vec{k}\cdot\vec{x}-\omega t)}$$

where $\vec{B}_0 = \hat{k}\times\vec{E}_0$

after using $\vec{B} = \vec{\nabla}\times\vec{A}$

Coulomb gauge condition $\vec{\nabla}\cdot\vec{A} = 0 \Longrightarrow \vec{k}\cdot\vec{A}_0 = 0$

Transverse waves: $\vec{k}\cdot\vec{E}_0 = \vec{k}\cdot\vec{B}_0 = 0$