The electromagnetic fields of a uniformly moving charge

1. Relativistic transformation laws

Consider a charge q moving at constant velocity $\vec{\boldsymbol{v}}$ with respect to the laboratory frame K. The rest frame of the charge will be denoted by K'. In particular, we define the origin of K' to be the location of the charge. A laboratory observer is located at the point $\vec{\boldsymbol{x}} = (x, y, z)$, which denotes the vector that points from the origin of the laboratory frame to the observer. As seen in the rest frame of the charge, the observer is located at the point $\vec{\boldsymbol{x}}\,' = (x', y', z')$, which denotes the vector that points from the origin of K' to the observer.

At time $t = 0$, the charge is located at the origin of the laboratory frame. After a time t has elapsed (as measured in frame K), the charge is located at the point $\vec{\boldsymbol{v}}t$ in the laboratory frame. It is convenient to define the axes of the K' coordinate system such that the K and K' coordinate systems (and their origins) coincide at $t = t' = 0$. As usual we define $x_0 \equiv ct$ and $x'_0 \equiv ct'$. The relation between $(x_0\,;\,\vec{\boldsymbol{x}})$ and $(x'_0\,;\,\vec{\boldsymbol{x}}\,')$ is given by

$$
\begin{aligned}
x'_0 &= \gamma(x_0 - \vec{\boldsymbol{\beta}}\boldsymbol{\cdot}\vec{\boldsymbol{x}})\,, \\
\vec{\boldsymbol{x}}\,' &= \vec{\boldsymbol{x}} + \frac{(\gamma-1)}{\beta^2}(\vec{\boldsymbol{\beta}}\boldsymbol{\cdot}\vec{\boldsymbol{x}})\vec{\boldsymbol{\beta}} - \gamma\vec{\boldsymbol{\beta}}x_0\,,
\end{aligned}
\tag{1}
$$

where

$$
\vec{\boldsymbol{\beta}} \equiv \vec{\boldsymbol{v}}/c\,, \qquad \beta \equiv |\vec{\boldsymbol{\beta}}|\,, \qquad \gamma \equiv \frac{1}{\sqrt{1-\beta^2}}\,.
$$

It is convenient to resolve all vectors into components parallel and perpendicular to the direction of the velocity $\hat{\boldsymbol{\beta}} \equiv \vec{\boldsymbol{\beta}}/\beta$. We shall write

$$
\vec{\boldsymbol{x}} = \vec{\boldsymbol{x}}_\parallel + \vec{\boldsymbol{x}}_\perp\,,
$$

where $\vec{\boldsymbol{x}}_\parallel \times \vec{\boldsymbol{\beta}} = \vec{\boldsymbol{x}}_\perp\boldsymbol{\cdot}\vec{\boldsymbol{\beta}} = 0$. Note that

$$
\vec{\boldsymbol{x}}_\parallel = \left(\frac{\vec{\boldsymbol{\beta}}\boldsymbol{\cdot}\vec{\boldsymbol{x}}}{\beta^2}\right)\vec{\boldsymbol{\beta}}\,,
$$

or equivalently,

$$
\vec{\boldsymbol{x}}_\parallel = x_\parallel\hat{\boldsymbol{\beta}}\,, \qquad \text{where } x_\parallel = \frac{\vec{\boldsymbol{\beta}}\boldsymbol{\cdot}\vec{\boldsymbol{x}}}{\beta}\,.
$$

Then eq. (1) can be rewritten as:

$$
\begin{aligned}
x'_0 &= \gamma(x_0 - \vec{\boldsymbol{\beta}}\boldsymbol{\cdot}\vec{\boldsymbol{x}}_\parallel)\,, \\
\vec{\boldsymbol{x}}\,'_\parallel &= \gamma(\vec{\boldsymbol{x}}_\parallel - \vec{\boldsymbol{\beta}}\,x_0)\,, \\
\vec{\boldsymbol{x}}\,'_\perp &= \vec{\boldsymbol{x}}_\perp\,.
\end{aligned}
\tag{2}
$$

The transformation laws for the electromagnetic fields are given by:

$$\vec{\boldsymbol{E}}' = \gamma(\vec{\boldsymbol{E}} + \vec{\boldsymbol{\beta}} \times \vec{\boldsymbol{B}}) - \frac{\gamma^2}{\gamma+1}\vec{\boldsymbol{\beta}}(\vec{\boldsymbol{\beta}}\boldsymbol{\cdot}\vec{\boldsymbol{E}})\,, \tag{3}$$

$$\vec{\boldsymbol{B}}' = \gamma(\vec{\boldsymbol{B}} - \vec{\boldsymbol{\beta}} \times \vec{\boldsymbol{E}}) - \frac{\gamma^2}{\gamma+1}\vec{\boldsymbol{\beta}}(\vec{\boldsymbol{\beta}}\boldsymbol{\cdot}\vec{\boldsymbol{B}})\,. \tag{4}$$

Again, we resolve the vectors into components parallel and perpendicular to the velocity,

$$\vec{\boldsymbol{E}} = \vec{\boldsymbol{E}}_\perp + \vec{\boldsymbol{E}}_\parallel\,, \qquad\qquad \vec{\boldsymbol{B}} = \vec{\boldsymbol{B}}_\perp + \vec{\boldsymbol{B}}_\parallel\,, \tag{5}$$

where

$$\vec{\boldsymbol{E}}_\parallel = \frac{\vec{\boldsymbol{\beta}}(\vec{\boldsymbol{\beta}}\boldsymbol{\cdot}\vec{\boldsymbol{E}})}{\beta^2}\,, \tag{6}$$

and

$$\vec{\boldsymbol{E}}_\perp = \vec{\boldsymbol{E}} - \vec{\boldsymbol{E}}_\parallel = \vec{\boldsymbol{E}} - \frac{\vec{\boldsymbol{\beta}}(\vec{\boldsymbol{\beta}}\boldsymbol{\cdot}\vec{\boldsymbol{E}})}{\beta^2} = \frac{\beta^2\vec{\boldsymbol{E}} - \vec{\boldsymbol{\beta}}(\vec{\boldsymbol{\beta}}\boldsymbol{\cdot}\vec{\boldsymbol{E}})}{\beta^2} = \frac{\vec{\boldsymbol{\beta}} \times (\vec{\boldsymbol{E}} \times \vec{\boldsymbol{\beta}})}{\beta^2}\,, \tag{7}$$

and similarly for $\vec{\boldsymbol{B}}_\parallel$ and $\vec{\boldsymbol{B}}_\perp$. Hence eq. (3) yields

$$\vec{\boldsymbol{\beta}}\boldsymbol{\cdot}\vec{\boldsymbol{E}}' = \gamma\vec{\boldsymbol{\beta}}\boldsymbol{\cdot}\vec{\boldsymbol{E}} - \frac{\gamma^2\beta^2}{\gamma+1}\,\vec{\boldsymbol{\beta}}\boldsymbol{\cdot}\vec{\boldsymbol{E}}\,. \tag{8}$$

Multiplying both sides of eq. (8) by $\vec{\boldsymbol{\beta}}/\beta^2$ and using eq. (6), it follows that

$$\vec{\boldsymbol{E}}'_\parallel = \left(\gamma - \frac{\gamma^2\beta^2}{\gamma+1}\right)\vec{\boldsymbol{E}}_\parallel = \vec{\boldsymbol{E}}_\parallel\,, \tag{9}$$

after noting that $\gamma^2 = 1/(1-\beta^2)$ yields $\gamma^2\beta^2 = (\gamma^2 - 1)$. Similarly,

$$\begin{aligned}\vec{\boldsymbol{E}}'_\perp = \frac{\vec{\boldsymbol{\beta}} \times (\vec{\boldsymbol{E}}' \times \vec{\boldsymbol{\beta}})}{\beta^2} &= \frac{\vec{\boldsymbol{\beta}}}{\beta^2} \times \left[\gamma(\vec{\boldsymbol{E}} + \vec{\boldsymbol{\beta}} \times \vec{\boldsymbol{B}}) \times \vec{\boldsymbol{\beta}}\right] \\ &= \frac{\gamma\vec{\boldsymbol{\beta}} \times (\vec{\boldsymbol{E}} \times \vec{\boldsymbol{\beta}})}{\beta^2} + \frac{\gamma\vec{\boldsymbol{\beta}} \times \left[\vec{\boldsymbol{\beta}} \times (\vec{\boldsymbol{B}} \times \vec{\boldsymbol{\beta}})\right]}{\beta^2} \\ &= \gamma\big(\vec{\boldsymbol{E}}_\perp + \vec{\boldsymbol{\beta}} \times \vec{\boldsymbol{B}}_\perp\big)\,. \end{aligned} \tag{10}$$

Repeating the analogous calculations for the magnetic fields, the end result is:

$$\vec{\boldsymbol{E}}'_\parallel = \vec{\boldsymbol{E}}_\parallel\,, \qquad\qquad \vec{\boldsymbol{E}}'_\perp = \gamma(\vec{\boldsymbol{E}}_\perp + \vec{\boldsymbol{\beta}} \times \vec{\boldsymbol{B}}_\perp)\,, \tag{11}$$

$$\vec{\boldsymbol{B}}'_\parallel = \vec{\boldsymbol{B}}_\parallel\,, \qquad\qquad \vec{\boldsymbol{B}}'_\perp = \gamma(\vec{\boldsymbol{B}}_\perp - \vec{\boldsymbol{\beta}} \times \vec{\boldsymbol{E}}_\perp)\,. \tag{12}$$

In analyzing the uniformly moving charge, $\vec{\boldsymbol{E}}'$ and $\vec{\boldsymbol{B}}'$ are known, so we have to invert eqs. (11) and (12) to obtain the electromagnetic fields in frame K. This is easily done by interchanging the primed and unprimed fields while reversing the sign of $\vec{\boldsymbol{\beta}}$. That is,

$$\vec{\boldsymbol{E}}_\perp = \gamma(\vec{\boldsymbol{E}}'_\perp - \vec{\boldsymbol{\beta}} \times \vec{\boldsymbol{B}}'_\perp)\,, \qquad\qquad \vec{\boldsymbol{B}}_\perp = \gamma(\vec{\boldsymbol{B}}'_\perp + \vec{\boldsymbol{\beta}} \times \vec{\boldsymbol{E}}'_\perp)\,. \tag{13}$$

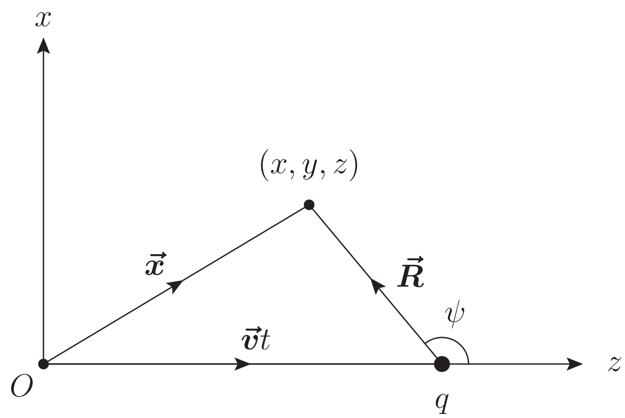

Figure 1: A charge q moving at constant velocity $\vec{\boldsymbol{v}}$ in the z-direction as seen from reference frame K. The origin of the laboratory frame K is denoted by O. The angle ψ is defined so that $\hat{\boldsymbol{v}}\boldsymbol{\cdot}\hat{\boldsymbol{R}} = \cos\psi$. By convention, we take $0 \leq \psi \leq \pi$.

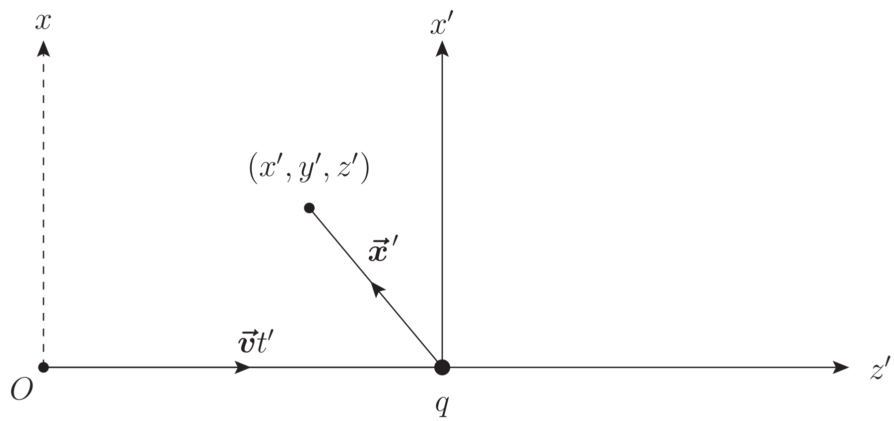

Figure 2: A charge q moving at constant velocity $\vec{\boldsymbol{v}}$ in the z'-direction as seen from reference frame K'. The origin of the laboratory frame K is denoted by O. The x-axis of frame K is indicated by a dashed line.

2. Electromagnetic fields of a uniformly moving charge[1]

Let us compare the views from reference frames K and K'. The moving charge as seen from the laboratory frame K is shown in Fig. 1.[2] In addition, we define $\vec{\boldsymbol{R}}$ to be the vector in frame K that points from the location of the charge at time t to the location of the observer. It follows that

$$\vec{\boldsymbol{R}} = \vec{\boldsymbol{x}} - \vec{\boldsymbol{v}}t\,. \tag{14}$$

The rest frame K' of the moving charge is depicted in Fig. 2. In this frame, the vector that points from the origin of frame K to the location of the charge is $\vec{\boldsymbol{v}}t'$, where t' is the time elapsed as measured in frame K' (where $t = t' = 0$ marks the time when the frames K and K' coincided).

[1]The derivation presented in these notes is inspired by Section 22.6.4 on pp. 844–845 of Andrew Zangwill, *Modern Electrodynamics* (Cambridge University Press, Cambridge, UK, 2013).

[2]In this figure, we have defined the z-axis to point in the direction of the velocity $\vec{\boldsymbol{v}}$, although we do not make us of this fact the derivation presented in these notes.

The goal of our calculation is to compute the electromagnetic fields, $\vec{\boldsymbol{E}}$ and $\vec{\boldsymbol{B}}$ in the laboratory frame K. First, we note that the corresponding electromagnetic fields in the rest frame K' of the charge are given (in gaussian units) by:

$$\vec{\boldsymbol{E}}\,' = \frac{q\vec{\boldsymbol{x}}\,'}{r'^3}\,, \qquad\qquad \vec{\boldsymbol{B}}\,' = 0\,, \tag{15}$$

where $r' \equiv |\vec{\boldsymbol{x}}\,'|$. We resolve the vectors above into components parallel and perpendicular to the velocity vector. In particular,

$$\vec{\boldsymbol{x}}\,' = \vec{\boldsymbol{x}}\,'_{\parallel} + \vec{\boldsymbol{x}}\,'_{\perp}\,,$$

and

$$r'^2 = \vec{\boldsymbol{x}}\,'\boldsymbol{\cdot}\vec{\boldsymbol{x}}\,' = \vec{\boldsymbol{x}}\,'_{\parallel}\boldsymbol{\cdot}\vec{\boldsymbol{x}}\,'_{\parallel} + \vec{\boldsymbol{x}}\,'_{\perp}\boldsymbol{\cdot}\vec{\boldsymbol{x}}\,'_{\perp}\,, \tag{16}$$

since $\vec{\boldsymbol{x}}_{\parallel}\boldsymbol{\cdot}\vec{\boldsymbol{x}}_{\perp} = 0$. Likewise, we can identify the longitudinal and transverse electric fields in frame K',

$$\vec{\boldsymbol{E}}\,'_{\parallel} = \frac{q\vec{\boldsymbol{x}}\,'_{\parallel}}{r'^3}\,, \qquad\qquad \vec{\boldsymbol{E}}\,'_{\perp} = \frac{q\vec{\boldsymbol{x}}\,'_{\perp}}{r'^3}\,. \tag{17}$$

We are now ready to evaluate the electromagnetic fields in frame K. First, we employ eqs. (11)–(13) and (15) to obtain

$$\vec{\boldsymbol{E}} = \vec{\boldsymbol{E}}_{\parallel} + \vec{\boldsymbol{E}}_{\perp} = \vec{\boldsymbol{E}}\,'_{\parallel} + \gamma\vec{\boldsymbol{E}}\,'_{\perp}\,, \qquad\qquad \vec{\boldsymbol{B}} = \vec{\boldsymbol{B}}_{\parallel} + \vec{\boldsymbol{B}}_{\perp} = \gamma\vec{\boldsymbol{\beta}}\times\vec{\boldsymbol{E}}\,'_{\perp}\,.$$

Using eq. (17), it follows that

$$\vec{\boldsymbol{E}} = \frac{q}{r'^3}\left(\vec{\boldsymbol{x}}\,'_{\parallel} + \gamma\vec{\boldsymbol{x}}\,'_{\perp}\right)\,, \qquad\qquad \vec{\boldsymbol{B}} = \frac{\gamma q}{r'^3}\,\vec{\boldsymbol{\beta}}\times\vec{\boldsymbol{x}}\,'_{\perp}\,.$$

We next apply eq. (2) to convert the primed coordinates into unprimed coordinates,

$$\vec{\boldsymbol{E}} = \frac{\gamma q}{r'^3}\left(\vec{\boldsymbol{x}}_{\parallel} - \vec{\boldsymbol{v}}t + \vec{\boldsymbol{x}}_{\perp}\right)\,, \qquad\qquad \vec{\boldsymbol{B}} = \frac{\gamma q}{cr'^3}\,\vec{\boldsymbol{v}}\times\vec{\boldsymbol{x}}_{\perp}\,, \tag{18}$$

after using $\vec{\boldsymbol{\beta}}x_0 = \vec{\boldsymbol{v}}t$. Furthermore [cf. eq. (16)],

$$r'^2 = \gamma^2(\vec{\boldsymbol{x}}_{\parallel} - \vec{\boldsymbol{v}}t)\boldsymbol{\cdot}(\vec{\boldsymbol{x}}_{\parallel} - \vec{\boldsymbol{v}}t) + \vec{\boldsymbol{x}}_{\perp}\boldsymbol{\cdot}\vec{\boldsymbol{x}}_{\perp}\,. \tag{19}$$

In frame K, we can also decompose the vector $\vec{\boldsymbol{R}}$ [cf. eq. (14)] into components parallel and perpendicular to the velocity vector. In particular,

$$\vec{\boldsymbol{R}}_{\parallel} = \vec{\boldsymbol{x}}_{\parallel} - \vec{\boldsymbol{v}}t\,, \qquad\qquad \vec{\boldsymbol{R}}_{\perp} = \vec{\boldsymbol{x}}_{\perp}\,.$$

Note that

$$\vec{\boldsymbol{x}}_{\parallel} - \vec{\boldsymbol{v}}t + \vec{\boldsymbol{x}}_{\perp} = \vec{\boldsymbol{R}}_{\parallel} + \vec{\boldsymbol{R}}_{\perp} = \vec{\boldsymbol{R}}\,, \qquad\qquad \vec{\boldsymbol{v}}\times\vec{\boldsymbol{x}}_{\perp} = \vec{\boldsymbol{v}}\times\vec{\boldsymbol{R}}_{\perp} = \vec{\boldsymbol{v}}\times\vec{\boldsymbol{R}}\,,$$

where the last result is obtained by noting that $\vec{\boldsymbol{v}}\times\vec{\boldsymbol{R}}_{\parallel} = 0$. Inserting the above results into eqs. (18) and (19) yields $r'^2 = \gamma^2 R_{\parallel}^2 + R_{\perp}^2$ and

$$\vec{\boldsymbol{E}} = \frac{\gamma q\vec{\boldsymbol{R}}}{(\gamma^2 R_{\parallel}^2 + R_{\perp}^2)^{3/2}}\,, \qquad\qquad \vec{\boldsymbol{B}} = \frac{\gamma q\vec{\boldsymbol{v}}\times\vec{\boldsymbol{R}}}{c(\gamma^2 R_{\parallel}^2 + R_{\perp}^2)^{3/2}}\,, \tag{20}$$

where $R_{\perp} \equiv |\vec{\boldsymbol{R}}_{\perp}|$ and $R_{\parallel} \equiv |\vec{\boldsymbol{R}}_{\parallel}|$. Note that $R \equiv |\vec{\boldsymbol{R}}| = (R_{\parallel}^2 + R_{\perp}^2)^{1/2}$.

As exhibited in Fig. 1, the angle between the vectors $\vec{\boldsymbol{v}}$ and $\vec{\boldsymbol{R}}$ is denoted by ψ. In particular, $R_\perp = R\sin\psi$. By convention, we take $0 \leq \psi \leq \pi$ (that is, ψ is a polar angle of $\vec{\boldsymbol{R}}$ with respect to the vector $\vec{\boldsymbol{v}}$). It follows that

$$\begin{aligned} r'^3 &= (\gamma^2 R_\parallel^2 + R_\perp^2)^{3/2} = \gamma^3(R_\parallel^2 + R_\perp^2/\gamma^2)^{3/2} = \gamma^3[R_\parallel^2 + R_\perp^2(1-\beta^2)]^{3/2} \\ &= \gamma^3(R^2 - R_\perp^2\beta^2)^{3/2} = \gamma^3 R^3(1-\beta^2\sin^2\psi)^{3/2}\,. \end{aligned} \tag{21}$$

Using eq. (21) in eq. (20), we arrive at our final result:

$$\vec{\boldsymbol{E}}(\vec{\boldsymbol{x}}) = \frac{q\vec{\boldsymbol{R}}}{\gamma^2 R^3(1-\beta^2\sin^2\psi)^{3/2}}\,, \qquad\qquad \vec{\boldsymbol{B}}(\vec{\boldsymbol{x}}) = \frac{q\,\vec{\boldsymbol{v}}\times\vec{\boldsymbol{R}}}{c\gamma^2 R^3(1-\beta^2\sin^2\psi)^{3/2}}\,, \tag{22}$$

where $\vec{\boldsymbol{R}} \equiv \vec{\boldsymbol{x}} - \vec{\boldsymbol{v}}t$. The expression for the electric field reproduces eq. (11.154) on p. 560 of Jackson.[3] However, our derivation is more general than the one given in Jackson, as Fig. 11.8 of Jackson assumes that the observer is located on the x-axis of Fig. 1; whereas in the derivation presented here the observer is located at an arbitrary point $\vec{\boldsymbol{x}} = (x, y, z)$.

3. Electromagnetic fields of a uniformly moving charge revisited

The electromagnetic fields given in eq. (22) are functions of $\vec{\boldsymbol{x}}$ and t that are expressed in terms of the variables R and ψ. In particular, the definitions of R and ψ are based on the location of the charge at the same time t [cf. Figure 1]. When we study the electromagnetic fields of a charge in general motion as in Chapter 14 of Jackson, the electromagnetic fields are expressed in terms of quantities whose definitions are based on the location of the charge at the *retarded time*, henceforth denoted by t'. By definition,[4]

$$t' = t - \frac{|\vec{\boldsymbol{x}} - \vec{\boldsymbol{r}}(t')|}{c}\,, \tag{23}$$

where $\vec{\boldsymbol{r}}(t') \equiv \vec{\boldsymbol{v}}t'$ is the location of the charge at time t'. The retarded time has the following interpretation: if a light signal originated from the location of the moving charge at time t', then the light signal would reach a fixed observer (located at the point $\vec{\boldsymbol{x}}$) at time t. It is instructive to rewrite eq. (22) in terms of quantities whose definitions are based on the location of the charge at the retarded time t'. The relevant quantities are defined in Figure 3.

[3]Jackson employs the symbol $\vec{\boldsymbol{r}}$ for what we call $\vec{\boldsymbol{R}}$. Our choice is motivated by the desire to avoid possible confusion between the meaning of the vector $\vec{\boldsymbol{x}}$ (with length $r \equiv |\vec{\boldsymbol{x}}|$) and the vector $\vec{\boldsymbol{R}}$ (with length R). In particular, it is important to note that although $\vec{\boldsymbol{R}}$ points from the charge to the observer in frame K and $\vec{\boldsymbol{x}}'$ points from the charge to the observer in frame K' [cf. Figs. 1 and 2], the vectors $\vec{\boldsymbol{R}}$ and $\vec{\boldsymbol{x}}'$ are *not* related by a Lorentz transformation. Of course, the coordinate vectors $\vec{\boldsymbol{x}}$ and $\vec{\boldsymbol{x}}'$ are related by a Lorentz transformation; namely a Lorentz boost along the direction of $\vec{\boldsymbol{v}}$ as indicated by eqs. (1) and (2).

[4]In Section 3, a primed variable indicate a variable that depends on the retarded time t'. In particular, primed variables in this section are *not* associated with the reference frame K' used in Sections 1 and 2. All computations in this section are performed in the laboratory frame K.

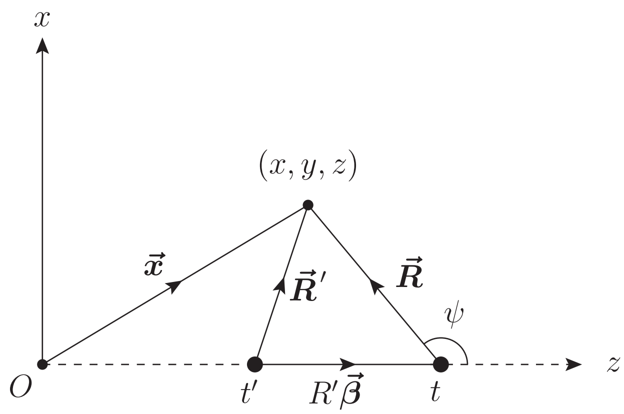

Figure 3: A charge q moving at constant velocity $\vec{\boldsymbol{v}}$ in the z-direction as seen from reference frame K. The origin of the laboratory frame K is denoted by O. The charge q is located at the origin at time $t = 0$. At the retarded time t', the charge q is located at $\vec{\boldsymbol{v}}t'$ (labeled by t'), and at time t, the charge is located at $\vec{\boldsymbol{v}}t$ (labeled by t). A fixed observer is located at the point $\vec{\boldsymbol{x}}$.

In particular, we have defined the vector $\vec{\boldsymbol{R}}' = \vec{\boldsymbol{x}} - \vec{\boldsymbol{r}}(t')$ to be the vector that points from the location of the charge at time t' to the location of the fixed observer. Thus, the retarded time given in eq. (23) can be rewritten as

$$t' = t - \frac{R'}{c}\,,$$

where $R' \equiv |\vec{\boldsymbol{R}}|$. Consequently, the vector that points from t' to t in Figure 3 is simply given by

$$\vec{\boldsymbol{v}}t - \vec{\boldsymbol{v}}t' = \frac{R'\vec{\boldsymbol{v}}}{c} = R'\vec{\boldsymbol{\beta}}\,.$$

It follows that

$$\vec{\boldsymbol{R}} = \vec{\boldsymbol{R}}' - R'\vec{\boldsymbol{\beta}} = R'(\hat{\boldsymbol{n}}' - \vec{\boldsymbol{\beta}})\,, \tag{24}$$

where $\hat{\boldsymbol{n}}' \equiv \vec{\boldsymbol{R}}'/R'$ is the unit vector that points in the direction of $\vec{\boldsymbol{R}}'$.

The angle ψ in Figure 3 can be defined via $\vec{\boldsymbol{R}} \boldsymbol{\cdot} \vec{\boldsymbol{\beta}} = R\beta\cos\psi$. It follows that

$$R^3(1-\beta^2\sin^2\psi)^{3/2} = R^3(1-\beta^2+\beta^2\cos^2\psi)^{3/2} = \left[R^2(1-\beta)^2 + (\vec{\boldsymbol{\beta}} \boldsymbol{\cdot} \vec{\boldsymbol{R}})^2\right]^{3/2}\,. \tag{25}$$

Using eq. (24), we can take the dot product with $\vec{\boldsymbol{\beta}}$ to obtain

$$\vec{\boldsymbol{\beta}} \boldsymbol{\cdot} \vec{\boldsymbol{R}} = R'(\hat{\boldsymbol{n}}' \boldsymbol{\cdot} \vec{\boldsymbol{\beta}} - \beta^2)\,,$$

after using $\vec{\boldsymbol{R}}' = R'\,\hat{\boldsymbol{n}}'$. Squaring eq. (24) yields

$$R^2 = R'^2\left(1+\beta^2 - 2\hat{\boldsymbol{n}}' \boldsymbol{\cdot} \vec{\boldsymbol{\beta}}\right)\,.$$

It follows that

$$\begin{aligned}
R^2(1-\beta)^2 + (\vec{\boldsymbol{\beta}} \boldsymbol{\cdot} \vec{\boldsymbol{R}})^2 &= R'^2\left[(1-\beta^2)(1+\beta^2-2\hat{\boldsymbol{n}}' \boldsymbol{\cdot} \vec{\boldsymbol{\beta}}) + (\hat{\boldsymbol{n}}' \boldsymbol{\cdot} \vec{\boldsymbol{\beta}} - \beta^2)^2\right] \\
&= R'^2\left[1-\beta^4 - 2\hat{\boldsymbol{n}}' \boldsymbol{\cdot} \vec{\boldsymbol{\beta}} + 2\beta^2\hat{\boldsymbol{n}}' \boldsymbol{\cdot} \vec{\boldsymbol{\beta}} + (\hat{\boldsymbol{n}}' \boldsymbol{\cdot} \vec{\boldsymbol{\beta}})^2 - 2\beta^2\hat{\boldsymbol{n}}' \boldsymbol{\cdot} \vec{\boldsymbol{\beta}} + \beta^4\right] \\
&= R'^2\left[1 - 2\hat{\boldsymbol{n}}' \boldsymbol{\cdot} \vec{\boldsymbol{\beta}} + (\hat{\boldsymbol{n}}' \boldsymbol{\cdot} \vec{\boldsymbol{\beta}})^2\right] = \left[R'(1-\hat{\boldsymbol{n}}' \boldsymbol{\cdot} \vec{\boldsymbol{\beta}})\right]^2\,.
\end{aligned}$$

Inserting this result into eq. (25), we end up with[5]

$$R^3(1-\beta^2\sin^2\psi)^{3/2} = R'^3(1-\hat{\boldsymbol{n}}'\cdot\vec{\boldsymbol{\beta}})^3\,. \tag{26}$$

Applying eqs. (24) and (26) to eq. (22), we obtain

$$\vec{\boldsymbol{E}}(\vec{\boldsymbol{x}}) = \frac{q(\hat{\boldsymbol{n}}'-\vec{\boldsymbol{\beta}})}{\gamma^2 R'^2(1-\hat{\boldsymbol{n}}'\cdot\vec{\boldsymbol{\beta}})^3}\,, \qquad\qquad \vec{\boldsymbol{B}}(\vec{\boldsymbol{x}}) = \frac{q\,\vec{\boldsymbol{\beta}}\times\hat{\boldsymbol{n}}'}{\gamma^2 R'^2(1-\hat{\boldsymbol{n}}'\cdot\vec{\boldsymbol{\beta}})^3} = \vec{\boldsymbol{n}}'\times\vec{\boldsymbol{E}}(\vec{\boldsymbol{x}})\,.$$

Note that all primed variables are functions of the retarded time t', since $\vec{\boldsymbol{R}}' \equiv \vec{\boldsymbol{x}} - \vec{\boldsymbol{r}}(t')$ and $\hat{\boldsymbol{n}}' \equiv \vec{\boldsymbol{R}}'/R'$. Thus, we have confirmed that the velocity fields obtained in eqs. (14.13) and (14.14) of Jackson are equivalent to the results obtained in eq. (22) by the Lorentz transformation technique of Section 2.

[5]Note that for massive particles, we have $0 \leq \beta < 1$ so that $1-\hat{\boldsymbol{n}}'\cdot\vec{\boldsymbol{\beta}} > 0$. Hence, it follows that $[(1-\hat{\boldsymbol{n}}'\cdot\vec{\boldsymbol{\beta}})^2]^{1/2} = 1-\hat{\boldsymbol{n}}'\cdot\vec{\boldsymbol{\beta}}$.