The Poisson equation and the inverse Laplacian

1. The Poisson equation and its solution

We wish to solve the Poisson equation,

$$\vec{\nabla}^2 \Phi = -4\pi\rho\,, \tag{1}$$

given a known charge distribution $\rho(\vec{r})$ that is nonzero over some finite volume of space, subject to boundary conditions (typically taken to be Dirichlet, in which Φ is specified over some closed surface or Neumann where $\vec{E} = -\vec{\nabla}\Phi$ is specified over some closed surface). The solution will take the form,

$$\Phi(\vec{r}) = \Phi_p(\vec{r}) + \Phi_c(\vec{r})\,, \tag{2}$$

where $\Phi_p(\vec{r})$ is a particular solution to the Poisson equation and $\Phi_c(\vec{r})$ is the (complementary) solution to the Laplace equation, $\vec{\nabla}^2\Phi_c(\vec{r}) = 0$. In defining the particular solution, we shall impose the condition that

$$\lim_{r\to\infty} \Phi_p(\vec{r}) = 0\,, \tag{3}$$

which can be viewed as a boundary condition that states that $\Phi_p(\vec{r})$ vanishes on the surface of a sphere of radius r in the limit of $r \to \infty$. Then

$$\Phi_p(\vec{r}) = \int \frac{\rho(\vec{r}\,')}{|\vec{r} - \vec{r}\,'|}\, d^3r'\,, \tag{4}$$

where the integration is taken over all of three-dimensional space. To prove that $\Phi_p(\vec{r})$ satisfies Poisson's equation subject to eq. (3), we first note that as $r \to \infty$, we have $|\vec{r} - \vec{r}\,'| = r[1 + \mathcal{O}(1/r)]$ so that

$$\lim_{r\to\infty} \Phi_p(\vec{r}) = \lim_{r\to\infty} \int \frac{\rho(\vec{r}\,')}{|\vec{r} - \vec{r}\,'|}\, d^3r' = \lim_{r\to\infty} \frac{1}{r}\int \rho(\vec{r}\,')d^3r' + \mathcal{O}\left(\frac{1}{r^2}\right) = \lim_{r\to\infty} \frac{q}{r} + \mathcal{O}\left(\frac{1}{r^2}\right) = 0\,, \tag{5}$$

where q is the total charge, under the assumption that the charge distribution is restricted to a finite region of space. Next, we compute the Laplacian of $\Phi_p(\vec{r})$,

$$\begin{aligned}\vec{\nabla}^2\Phi_p(\vec{r}) &= \vec{\nabla}^2 \int \frac{\rho(\vec{r}\,')}{|\vec{r} - \vec{r}\,'|}\, d^3r' = \int \rho(\vec{r}\,')\vec{\nabla}^2\left(\frac{1}{|\vec{r} - \vec{r}\,'|}\right) d^3r' \\ &= -4\pi \int \rho(\vec{r}\,')\delta^3(\vec{r} - \vec{r}\,')d^3r' = -4\pi\rho(\vec{r})\,, \end{aligned} \tag{6}$$

where we have used the well-known result,

$$\vec{\boldsymbol{\nabla}}^{\mathbf{2}}\left(\frac{1}{|\vec{\boldsymbol{r}}-\vec{\boldsymbol{r}}'|}\right)=-4\pi\delta^3(\vec{\boldsymbol{r}}-\vec{\boldsymbol{r}}')\,. \tag{7}$$

Note that $\vec{\boldsymbol{\nabla}}^{\mathbf{2}}$ involves derivatives with respect to $\vec{\boldsymbol{r}}$, so that in applying the Laplacian, the variable $\vec{\boldsymbol{r}}'$ (which is a dummy integration variable) is treated as being fixed. Thus, we have verified that $\Phi_p(\vec{\boldsymbol{r}})$ is a solution to Poisson's equation.

Indeed, $\Phi_p(\vec{\boldsymbol{r}})$ is the unique solution to Poisson's equation, which is valid at all points in space, subject to eq. (3). More general boundary value problems would involve solving Poisson's equation in a restricted region of space, V. In this case, we must specify the boundary conditions on the closed surface S of V. The solution is then given by:

$$\Phi(\vec{\boldsymbol{r}})=\Phi_c(\vec{\boldsymbol{r}})+\int_V\frac{\rho(\vec{\boldsymbol{r}}')}{|\vec{\boldsymbol{r}}-\vec{\boldsymbol{r}}'|}\,d^3r'\,, \tag{8}$$

where $\Phi_c(\vec{\boldsymbol{r}})$ is a solution to Laplace's equation, which is chosen such that the boundary conditions are satisfied when applied to the *complete* solution to the problem, $\Phi(\vec{\boldsymbol{r}})$.

2. The inverse Laplacian and the Green function

Consider the solution to Poisson's equation, which is valid at all points in space, subject to eq. (3),

$$\vec{\boldsymbol{\nabla}}^{\mathbf{2}}\Phi(\vec{\boldsymbol{r}})=-4\pi\rho(\vec{\boldsymbol{r}})\,, \tag{9}$$

where $\rho(\vec{\boldsymbol{r}})$ is nonzero only over some finite region in space. In fact, this last assumption is stronger than is necessary. It is sufficient to assume that $\rho(\vec{\boldsymbol{r}})\to 0$ as $r\to\infty$ fast enough such that the volume integral of $\rho(\vec{\boldsymbol{r}})$ over all space converges. Then, as discussed in Section 1, the solution to Poisson's equation is unique. That is, the solution to Poisson's equation is given by eq. (8) with $\Phi_c(\vec{\boldsymbol{r}})=0$. Equivalently, $\Phi(\vec{\boldsymbol{r}})=\Phi_p(\vec{\boldsymbol{r}})$, where $\Phi_p(\vec{\boldsymbol{r}})$ is given by eq. (4).

Under the stated conditions above, it is tempting to derive the solution to Poisson's equation by introducing the inverse Laplacian, $\vec{\boldsymbol{\nabla}}^{-\mathbf{2}}$. Operating with the inverse Laplacian on eq. (9) yields,

$$\vec{\boldsymbol{\nabla}}^{-\mathbf{2}}\vec{\boldsymbol{\nabla}}^{\mathbf{2}}\Phi(\vec{\boldsymbol{r}})=-4\pi\vec{\boldsymbol{\nabla}}^{-\mathbf{2}}\rho(\vec{\boldsymbol{r}})\,.$$

Clearly, one should define $\vec{\boldsymbol{\nabla}}^{-\mathbf{2}}\vec{\boldsymbol{\nabla}}^{\mathbf{2}}$ to be the identity operator, in which case we would conclude that

$$\Phi(\vec{\boldsymbol{r}})=-4\pi\vec{\boldsymbol{\nabla}}^{-\mathbf{2}}\rho(\vec{\boldsymbol{r}})\,. \tag{10}$$

Comparing this with $\Phi(\vec{\boldsymbol{r}})=\Phi_p(\vec{\boldsymbol{r}})$, where $\Phi_p(\vec{\boldsymbol{r}})$ is given by eq. (4), it follows that we should identify

$$\boxed{\vec{\boldsymbol{\nabla}}^{-\mathbf{2}}\rho(\vec{\boldsymbol{r}})=-\frac{1}{4\pi}\int\frac{\rho(\vec{\boldsymbol{r}}')}{|\vec{\boldsymbol{r}}-\vec{\boldsymbol{r}}'|}\,d^3r'} \tag{11}$$

Plugging eq. (11) back into eq. (10) yields

$$\Phi(\vec{\boldsymbol{r}}) = \int \frac{\rho(\vec{\boldsymbol{r}}')}{|\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}'|}\, d^3r' \,, \tag{12}$$

as expected.

The definition of the inverse Laplacian given in eq. (11) shows that this operator acts nonlocally. That is, the value of $\vec{\boldsymbol{\nabla}}^{-2}\rho(\vec{\boldsymbol{r}})$ at the point $\vec{\boldsymbol{r}}$ depends on $\rho(\vec{\boldsymbol{r}}')$ evaluated at all points in space. This should not be surprising to you. After all, the antiderivative of calculus is an integral! More importantly, the definition of the inverse Laplacian requires an assumption about the space of functions on which it acts. In the present case, we have required that the space of functions should only include twice differentiable functions that vanish sufficiently fast at infinity. To check that the definition of the inverse Laplacian given in eq. (11) is sensible, we perform the following two computations:

$$\begin{aligned}
\vec{\boldsymbol{\nabla}}^2\vec{\boldsymbol{\nabla}}^{-2}\rho(\vec{\boldsymbol{r}}) &= -\frac{1}{4\pi}\vec{\boldsymbol{\nabla}}^2 \int \frac{\rho(\vec{\boldsymbol{r}}')}{|\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}'|}\, d^3r' = -\frac{1}{4\pi}\int \rho(\vec{\boldsymbol{r}}')\vec{\boldsymbol{\nabla}}^2\left(\frac{1}{|\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}'|}\right) d^3r' \\
&= \int \rho(\vec{\boldsymbol{r}}')\delta^3(\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}')d^3r' = \rho(\vec{\boldsymbol{r}}) \,,
\end{aligned}$$

and

$$\begin{aligned}
\vec{\boldsymbol{\nabla}}^{-2}\left[\vec{\boldsymbol{\nabla}}^2\rho(\vec{\boldsymbol{r}})\right] &= -\frac{1}{4\pi}\int \frac{\vec{\boldsymbol{\nabla}}'^2\rho(\vec{\boldsymbol{r}}')}{|\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}'|}\, d^3r' = -\frac{1}{4\pi}\int \rho(\vec{\boldsymbol{r}}')\vec{\boldsymbol{\nabla}}'^2\left(\frac{1}{|\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}'|}\right) d^3r' \\
&= \int \rho(\vec{\boldsymbol{r}}')\delta^3(\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}')d^3r' = \rho(\vec{\boldsymbol{r}}) \,,
\end{aligned} \tag{13}$$

where $\vec{\boldsymbol{\nabla}}'^2$ is the Laplacian that involves derivatives with respect to $\vec{\boldsymbol{r}}'$. Note that in deriving eq. (13) we integrated by parts twice using Green's second identity (details are provided in Appendix A). In particular, we took $\psi = \rho$ and $\phi = 1/|\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}'|$ in eq. (26). Indeed, as both $\rho(\vec{\boldsymbol{r}}')$ and $1/|\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}'|$ vanish at the surface of infinity, we are justified in setting the right hand side of eq. (25) to zero. Thus, we have shown that eq. (11), subject to restrictions on $\rho(\vec{\boldsymbol{r}})$ at infinity, satisfies

$$\vec{\boldsymbol{\nabla}}^2\vec{\boldsymbol{\nabla}}^{-2}\rho(\vec{\boldsymbol{r}}) = \vec{\boldsymbol{\nabla}}^{-2}\vec{\boldsymbol{\nabla}}^2\rho(\vec{\boldsymbol{r}}) = \rho(\vec{\boldsymbol{r}}) \,,$$

which confirms that both $\vec{\boldsymbol{\nabla}}^2\vec{\boldsymbol{\nabla}}^{-2}$ and $\vec{\boldsymbol{\nabla}}^2\vec{\boldsymbol{\nabla}}^{-2}$ are equivalent to the identity operator.

The inverse Laplacian can also be used to determine the Green function of Poisson's equation. First we assume that the potential vanishes sufficiently fast at infinity, as discussed below eq. (3). We define the Green function $G(\vec{\boldsymbol{r}}, \vec{\boldsymbol{r}}')$ to be the solution of

$$\vec{\boldsymbol{\nabla}}^2 G(\vec{\boldsymbol{r}}, \vec{\boldsymbol{r}}') = -4\pi\delta^3(\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}') \,. \tag{14}$$

Then,

$$G(\vec{\boldsymbol{r}},\vec{\boldsymbol{r}}') = -4\pi\vec{\boldsymbol{\nabla}}^{-2}\delta^3(\vec{\boldsymbol{r}}-\vec{\boldsymbol{r}}') = \int \frac{\delta^3(\vec{\boldsymbol{r}}''-\vec{\boldsymbol{r}}')}{|\vec{\boldsymbol{r}}-\vec{\boldsymbol{r}}''|}d^3r'' = \frac{1}{|\vec{\boldsymbol{r}}-\vec{\boldsymbol{r}}'|}\,. \tag{15}$$

Thus, the inverse Laplacian provides a very quick derivation of the Green function. The interpretation of the Green function is clear—it is the potential that arises due to the presence of a point charge located at $\vec{\boldsymbol{r}}'$. The utility of the Green function is that it can be used to construct the potential for an arbitrary charge density via

$$\Phi(\vec{\boldsymbol{r}}) = \int G(\vec{\boldsymbol{r}},\vec{\boldsymbol{r}}')\rho(\vec{\boldsymbol{r}}')d^3r'\,, \tag{16}$$

since eq. (16) implies that $\Phi(\vec{\boldsymbol{r}})$ satisfies the Poisson equation, i.e.,

$$\vec{\boldsymbol{\nabla}}^2\Phi(\vec{\boldsymbol{r}}) = \int \rho(\vec{\boldsymbol{r}})\vec{\boldsymbol{\nabla}}^2 G(\vec{\boldsymbol{r}},\vec{\boldsymbol{r}}') = -4\pi\int \rho(\vec{\boldsymbol{r}})\delta^3(\vec{\boldsymbol{r}}-\vec{\boldsymbol{r}}') = -4\pi\rho(\vec{\boldsymbol{r}})\,.$$

Another interpretation of the Green function can be ascertained from eq. (15). The Dirac delta function is the function space analog of the Kronecker delta δ_{ij}. Thus, the Dirac delta function is an infinite dimensional matrix corresponding to the identity matrix, where $\delta^3(\vec{\boldsymbol{r}}-\vec{\boldsymbol{r}}')$ are the matrix elements of this infinite dimensional matrix. Apart from the overall factor of -4π (which is a matter of convention), $G(\vec{\boldsymbol{r}},\vec{\boldsymbol{r}}')$ are the matrix elements of the infinite dimensional matrix that represents the inverse Laplacian.

In more general boundary value problems, one must solve the Poisson equation in a restricted region of space, V. In this case, we must specify the boundary conditions on the closed surface S of V. The corresponding Green function is still a solution to eq. (14), but it now must also satisfy the relevant boundary conditions. Thus, in analogy to eq. (8), the Green function takes the form

$$G(\vec{\boldsymbol{r}},\vec{\boldsymbol{r}}') = F(\vec{\boldsymbol{r}},\vec{\boldsymbol{r}}') + \frac{1}{|\vec{\boldsymbol{r}}-\vec{\boldsymbol{r}}'|}\,, \tag{17}$$

where $F(\vec{\boldsymbol{r}},\vec{\boldsymbol{r}}')$ is a solution to the Laplace equation that is adjusted in order that $G(\vec{\boldsymbol{r}},\vec{\boldsymbol{r}}')$ satisfy the relevant boundary conditions. In this case, eq. (16) yields

$$\Phi(\vec{\boldsymbol{r}}) = \int_V G(\vec{\boldsymbol{r}},\vec{\boldsymbol{r}}')\rho(\vec{\boldsymbol{r}}')d^3r' = \int_V F(\vec{\boldsymbol{r}},\vec{\boldsymbol{r}}')\rho(\vec{\boldsymbol{r}}')d^3r' + \int_V \frac{\rho(\vec{\boldsymbol{r}}')}{|\vec{\boldsymbol{r}}-\vec{\boldsymbol{r}}'|}d^3r'\,.$$

Comparing with eq. (8), we identify

$$\Phi_c(\vec{\boldsymbol{r}}) = \int_V F(\vec{\boldsymbol{r}},\vec{\boldsymbol{r}}')\rho(\vec{\boldsymbol{r}}')d^3r'\,.$$

The inverse Laplacian was defined in eq. (11) under the assumption that it acts on functions (defined at all points in space) that vanish sufficiently fast at infinity.

In contrast, if the functions are defined only in a restricted region of space V, then the inverse Laplacian is ill-defined unless one imposes boundary conditions on the closed surface S of V. This can be understood as follows. If we solve the Laplace equation inside V, we find non-trivial solutions, denoted by $\Phi_c(\vec{\boldsymbol{r}})$ in eq. (2). That is, the Laplacian possesses an eigenfunction $\Phi_c(\vec{\boldsymbol{r}})$ with corresponding eigenvalue equal to zero. This immediately implies that $\vec{\boldsymbol{\nabla}}^{-2}$ is ill-defined; otherwise one would obtain eq. (11) instead of the correct result given in eq. (8).[1] This means that eq. (10) does not determine $\Phi(\vec{\boldsymbol{r}})$ uniquely. This is not surprising, as we have not yet specified the boundary conditions on S. However, once we specify these conditions, $\Phi(\vec{\boldsymbol{r}})$ is uniquely determined. This means that the definition of $\vec{\boldsymbol{\nabla}}^{-2}$ [which generalizes eq. (11)] becomes well-defined. This is not surprising, since we know that the form of the Green function depends in detail on the boundary conditions that are applied, which determines $F(\vec{\boldsymbol{r}},\vec{\boldsymbol{r}}\,')$ as indicated in eq. (17).

3. An application of the inverse Laplacian

In this section, we provide an interesting application of the inverse Laplacian in proving the Helmholtz decomposition of a vector field $\vec{\boldsymbol{V}}(\vec{\boldsymbol{r}})$ that exists in all of space. We assume that $\vec{\boldsymbol{V}}(\vec{\boldsymbol{r}})$ vanishes sufficiently fast as $r \to \infty$. The Helmholtz theorem states that the following decomposition is unique,

$$\vec{\boldsymbol{V}} = \vec{\boldsymbol{V}}_{\mathbf{tr}} + \vec{\boldsymbol{V}}_{\mathbf{long}}\,, \qquad \text{where } \vec{\boldsymbol{\nabla}}\boldsymbol{\cdot}\vec{\boldsymbol{V}}_{\mathbf{tr}} = 0 \text{ and } \vec{\boldsymbol{\nabla}} \times \vec{\boldsymbol{V}}_{\mathbf{long}} = 0\,. \tag{18}$$

Any vector $\vec{\boldsymbol{V}}_{\mathbf{tr}}$ that satisfies $\vec{\boldsymbol{\nabla}}\boldsymbol{\cdot}\vec{\boldsymbol{V}}_{\mathbf{tr}} = 0$ is called solenoidal or transverse. Any vector $\vec{\boldsymbol{V}}_{\mathbf{long}}$ that satisfies $\vec{\boldsymbol{\nabla}} \times \vec{\boldsymbol{V}}_{\mathbf{long}} = 0$ is called irrotational or longitudinal.[2]

To prove that the decomposition given in eq. (18) exists, we can directly construct $\vec{\boldsymbol{V}}_{\mathbf{tr}}$ and $\vec{\boldsymbol{V}}_{\mathbf{long}}$ with the help of the inverse Laplacian. I claim that

$$\vec{\boldsymbol{V}}_{\mathbf{long}} = \vec{\boldsymbol{\nabla}}\vec{\boldsymbol{\nabla}}^{-2}(\vec{\boldsymbol{\nabla}}\boldsymbol{\cdot}\vec{\boldsymbol{V}})\,. \tag{19}$$

First, we observe that $\vec{\boldsymbol{V}}_{\mathbf{long}} = \vec{\boldsymbol{\nabla}}\psi$, where $\psi \equiv \vec{\boldsymbol{\nabla}}^{-2}(\vec{\boldsymbol{\nabla}}\boldsymbol{\cdot}\vec{\boldsymbol{V}})$. Thus, it follows that $\vec{\boldsymbol{\nabla}}\times\vec{\boldsymbol{V}}_{\mathbf{long}} = \vec{\boldsymbol{\nabla}}\times\vec{\boldsymbol{\nabla}}\psi = 0$, as required. Next, we use eqs. (18) and (19) to compute $\vec{\boldsymbol{V}}_{\mathbf{tr}}$,

$$\vec{\boldsymbol{V}}_{\mathbf{tr}} = \vec{\boldsymbol{V}} - \vec{\boldsymbol{V}}_{\mathbf{long}} = \vec{\boldsymbol{V}} - \vec{\boldsymbol{\nabla}}\vec{\boldsymbol{\nabla}}^{-2}(\vec{\boldsymbol{\nabla}}\boldsymbol{\cdot}\vec{\boldsymbol{V}})\,. \tag{20}$$

We now check that

$$\vec{\boldsymbol{\nabla}}\boldsymbol{\cdot}\vec{\boldsymbol{V}}_{\mathbf{tr}} = \vec{\boldsymbol{\nabla}}\boldsymbol{\cdot}\left[\vec{\boldsymbol{V}} - \vec{\boldsymbol{\nabla}}\vec{\boldsymbol{\nabla}}^{-2}(\vec{\boldsymbol{\nabla}}\boldsymbol{\cdot}\vec{\boldsymbol{V}})\right] = \vec{\boldsymbol{\nabla}}\boldsymbol{\cdot}\vec{\boldsymbol{V}} - \vec{\boldsymbol{\nabla}}^{2}\vec{\boldsymbol{\nabla}}^{-2}(\vec{\boldsymbol{\nabla}}\boldsymbol{\cdot}\vec{\boldsymbol{V}}) = \vec{\boldsymbol{\nabla}}\boldsymbol{\cdot}\vec{\boldsymbol{V}} - \vec{\boldsymbol{\nabla}}\boldsymbol{\cdot}\vec{\boldsymbol{V}} = 0\,,$$

as required. This completes the proof that the Helmholtz decomposition exists.

[1] Consider the analogous case of a finite dimensional operator and its matrix representation M. If M has a zero eigenvalue, then its determinant vanishes (since $\det M$ is the product of its eigenvalues), in which case M^{-1} is ill-defined.

[2] The terminology transverse and longitudinal implicit in eq. (18) arises from the study of vector waves of the form $\vec{\boldsymbol{V}}(\vec{\boldsymbol{r}},t) = \vec{\boldsymbol{E}}_0 e^{i\vec{\boldsymbol{k}}\boldsymbol{\cdot}\vec{\boldsymbol{r}} - i\omega t}$. Then, $\vec{\boldsymbol{\nabla}}\boldsymbol{\cdot}\vec{\boldsymbol{V}}_{\mathbf{tr}} = 0$ implies that $\vec{\boldsymbol{k}}\boldsymbol{\cdot}\vec{\boldsymbol{V}}_{\mathbf{tr}} = 0$, which means that $\vec{\boldsymbol{V}}_{\mathbf{tr}}$ is *transverse* to the direction of the wave (which propagates along $\vec{\boldsymbol{k}}$). Likewise, $\vec{\boldsymbol{\nabla}} \times \vec{\boldsymbol{V}}_{\mathbf{long}} = 0$ implies that $\vec{\boldsymbol{k}}\times\vec{\boldsymbol{V}}_{\mathbf{long}} = 0$, which means that $\vec{\boldsymbol{V}}_{\mathbf{long}}$ is *longitudinal*, i.e. parallel to the direction of the wave.

Using the definition of the inverse Laplacian given by eq. (11), it follows from eq. (19) that

$$\vec{\boldsymbol{V}}_{\mathbf{long}}(\vec{\boldsymbol{r}}) = -\frac{1}{4\pi}\vec{\boldsymbol{\nabla}}\int \frac{\vec{\boldsymbol{\nabla}}'\boldsymbol{\cdot}\vec{\boldsymbol{V}}(\vec{\boldsymbol{r}}')}{|\vec{\boldsymbol{r}}-\vec{\boldsymbol{r}}'|}\,d^3r'\,,$$

where the integral is taken over all space. We can also determine $\vec{\boldsymbol{V}}_{\mathbf{tr}}$ from eq. (20)

$$\vec{\boldsymbol{V}}_{\mathbf{tr}}(\vec{\boldsymbol{r}}) = \vec{\boldsymbol{V}}(\vec{\boldsymbol{r}}) - \vec{\boldsymbol{V}}_{\mathbf{long}}(\vec{\boldsymbol{r}}) = \vec{\boldsymbol{V}}(\vec{\boldsymbol{r}}) + \frac{1}{4\pi}\vec{\boldsymbol{\nabla}}\int \frac{\vec{\boldsymbol{\nabla}}'\boldsymbol{\cdot}\vec{\boldsymbol{V}}(\vec{\boldsymbol{r}}')}{|\vec{\boldsymbol{r}}-\vec{\boldsymbol{r}}'|}\,d^3r'\,. \tag{21}$$

In Appendix B [cf. eq. (35)], we demonstrate that an equivalent expression for $\vec{\boldsymbol{V}}_{\mathbf{tr}}$ is given by:

$$\vec{\boldsymbol{V}}_{\mathbf{tr}} = -\vec{\boldsymbol{\nabla}}\times\vec{\boldsymbol{\nabla}}^{-2}(\vec{\boldsymbol{\nabla}}\times\vec{\boldsymbol{V}})\,. \tag{22}$$

which implies that an equivalent form to eq. (21) is given by:

$$\vec{\boldsymbol{V}}_{\mathbf{tr}}(\vec{\boldsymbol{r}}) = \frac{1}{4\pi}\vec{\boldsymbol{\nabla}}\times\int \frac{\vec{\boldsymbol{\nabla}}'\times\vec{\boldsymbol{V}}(\vec{\boldsymbol{r}}')}{|\vec{\boldsymbol{r}}-\vec{\boldsymbol{r}}'|}\,d^3r'\,. \tag{23}$$

Hence, the Helmholtz decomposition of a vector field, defined over all space, is unique and is given by

$$\vec{\boldsymbol{V}} = \vec{\boldsymbol{\nabla}}\vec{\boldsymbol{\nabla}}^{-2}(\vec{\boldsymbol{\nabla}}\boldsymbol{\cdot}\vec{\boldsymbol{V}}) - \vec{\boldsymbol{\nabla}}\times\vec{\boldsymbol{\nabla}}^{-2}(\vec{\boldsymbol{\nabla}}\times\vec{\boldsymbol{V}})\,,$$

or equivalently by:

$$\vec{\boldsymbol{V}}(\vec{\boldsymbol{r}}) = -\frac{1}{4\pi}\vec{\boldsymbol{\nabla}}\int \frac{\vec{\boldsymbol{\nabla}}'\boldsymbol{\cdot}\vec{\boldsymbol{V}}(\vec{\boldsymbol{r}}')}{|\vec{\boldsymbol{r}}-\vec{\boldsymbol{r}}'|}\,d^3r' + \frac{1}{4\pi}\vec{\boldsymbol{\nabla}}\times\int \frac{\vec{\boldsymbol{\nabla}}'\times\vec{\boldsymbol{V}}(\vec{\boldsymbol{r}}')}{|\vec{\boldsymbol{r}}-\vec{\boldsymbol{r}}'|}\,d^3r'\,, \tag{24}$$

where the integration is taken over all space. It should be emphasized that eq. (24) is an *identity* for any vector field $\vec{\boldsymbol{V}}(\vec{\boldsymbol{r}})$ that vanishes sufficiently fast at infinity.

Another version of the Helmholtz decomposition states that under the same conditions as indicated above, $\vec{\boldsymbol{V}}(\vec{\boldsymbol{r}})$ can be rewritten in the form:

$$\vec{\boldsymbol{V}}(\vec{\boldsymbol{r}}) = -\vec{\boldsymbol{\nabla}}\Phi(\vec{\boldsymbol{r}}) + \vec{\boldsymbol{\nabla}}\times\vec{\boldsymbol{A}}(\vec{\boldsymbol{r}})\,.$$

Using eq. (24), we can read off the functions $\Phi(\vec{\boldsymbol{r}})$ and $\vec{\boldsymbol{A}}(\vec{\boldsymbol{r}})$,

$$\Phi(\vec{\boldsymbol{r}}) = \frac{1}{4\pi}\int \frac{\vec{\boldsymbol{\nabla}}'\boldsymbol{\cdot}\vec{\boldsymbol{V}}(\vec{\boldsymbol{r}}')}{|\vec{\boldsymbol{r}}-\vec{\boldsymbol{r}}'|}\,d^3r'\,, \qquad \vec{\boldsymbol{A}}(\vec{\boldsymbol{r}}) = \frac{1}{4\pi}\int \frac{\vec{\boldsymbol{\nabla}}'\times\vec{\boldsymbol{V}}(\vec{\boldsymbol{r}}')}{|\vec{\boldsymbol{r}}-\vec{\boldsymbol{r}}'|}\,d^3r'\,.$$

For example, if $\vec{\boldsymbol{V}}(\vec{\boldsymbol{r}}) = \vec{\boldsymbol{E}}(\vec{\boldsymbol{r}})$ is the electrostatic field, then $\vec{\boldsymbol{\nabla}}\times\vec{\boldsymbol{E}} = 0$ so that $\vec{\boldsymbol{A}}(\vec{\boldsymbol{r}}) = 0$. Using $\vec{\boldsymbol{\nabla}}\boldsymbol{\cdot}\vec{\boldsymbol{E}} = 4\pi\rho$, the above result for $\Phi(\vec{\boldsymbol{r}})$ reduces to eq. (12) as expected.

Appendix A: Green's second identity

Green's second identity is given by:

$$\int_V (\phi\vec{\boldsymbol{\nabla}}^{\mathbf{2}}\psi - \psi\vec{\boldsymbol{\nabla}}^{\mathbf{2}}\phi)d^3r = \oint_S (\phi\vec{\boldsymbol{\nabla}}\psi - \psi\vec{\boldsymbol{\nabla}}\phi)\boldsymbol{\cdot}\hat{\boldsymbol{n}}da\,. \tag{25}$$

If V is the volume of all space, then S is the surface at infinity. In many applications, one can show that the integrand $\phi\vec{\boldsymbol{\nabla}}\psi - \psi\vec{\boldsymbol{\nabla}}\phi$ vanishes at the surface of infinity. In this case, the right hand side of eq. (25) vanishes, and it follows that

$$\int \phi\vec{\boldsymbol{\nabla}}^{\mathbf{2}}\psi\, d^3r = \int \psi\vec{\boldsymbol{\nabla}}^{\mathbf{2}}\phi\, d^3r\,. \tag{26}$$

Appendix B: Proof of eq. (22)

In this appendix, I provide additional details to the discussion of the Helmholtz decomposition, $\vec{\boldsymbol{V}} = \vec{\boldsymbol{V}}_{\mathbf{tr}} + \vec{\boldsymbol{V}}_{\mathbf{long}}$ [cf. eq. (18)]. In Section 3, we showed that

$$\vec{\boldsymbol{V}}_{\mathbf{long}} = \vec{\boldsymbol{\nabla}}\vec{\boldsymbol{\nabla}}^{\mathbf{-2}}(\vec{\boldsymbol{\nabla}}\boldsymbol{\cdot}\vec{\boldsymbol{V}})\,, \qquad \vec{\boldsymbol{V}}_{\mathbf{tr}} = \vec{\boldsymbol{V}} - \vec{\boldsymbol{V}}_{\mathbf{long}}\,. \tag{27}$$

Here, I will derive an alternative expression for $\vec{\boldsymbol{V}}_{\mathbf{tr}}$,

$$\vec{\boldsymbol{V}}_{\mathbf{tr}} = -\vec{\boldsymbol{\nabla}}\times\vec{\boldsymbol{\nabla}}^{\mathbf{-2}}(\vec{\boldsymbol{\nabla}}\times\vec{\boldsymbol{V}})\,. \tag{28}$$

To derive this result, we need to establish two identities. First, for any vector function $\vec{\boldsymbol{A}}(\vec{\boldsymbol{r}})$ that vanishes sufficiently fast at infinity,

$$\vec{\boldsymbol{\nabla}}\times\vec{\boldsymbol{\nabla}}^{\mathbf{-2}}(\vec{\boldsymbol{A}}) = \vec{\boldsymbol{\nabla}}^{\mathbf{-2}}(\vec{\boldsymbol{\nabla}}\times\vec{\boldsymbol{A}})\,. \tag{29}$$

To prove this result, we employ the definition of the inverse Laplacian given in eq. (11), which yields[3]

$$-4\pi\vec{\boldsymbol{\nabla}}\times\vec{\boldsymbol{\nabla}}^{\mathbf{-2}}(\vec{\boldsymbol{A}}) = \vec{\boldsymbol{\nabla}}\times\int\frac{\vec{\boldsymbol{A}}(\vec{\boldsymbol{r}}')}{|\vec{\boldsymbol{r}}-\vec{\boldsymbol{r}}'|}d^3r' = -\int\vec{\boldsymbol{A}}(\vec{\boldsymbol{r}}')\times\vec{\boldsymbol{\nabla}}\left(\frac{1}{|\vec{\boldsymbol{r}}-\vec{\boldsymbol{r}}'|}\right)d^3r'\,. \tag{31}$$

[3]In obtaining eq. (31), we have used

$$\vec{\boldsymbol{\nabla}}\times\left(\psi\vec{\boldsymbol{B}}\right) = \psi\vec{\boldsymbol{\nabla}}\times\vec{\boldsymbol{B}} - \vec{\boldsymbol{B}}\times\vec{\boldsymbol{\nabla}}\psi\,. \tag{30}$$

In the application to eq. (31), $\psi\equiv 1/|\vec{\boldsymbol{r}}-\vec{\boldsymbol{r}}'|$ and $\vec{\boldsymbol{B}}\equiv\vec{\boldsymbol{A}}(\vec{\boldsymbol{r}}')$. Since the latter is independent of $\vec{\boldsymbol{r}}$, it follows that $\vec{\boldsymbol{\nabla}}\times\vec{\boldsymbol{B}} = 0$.

Noting that

$$\vec{\boldsymbol{\nabla}}\left(\frac{1}{|\vec{\boldsymbol{r}}-\vec{\boldsymbol{r}}'|}\right) = -\vec{\boldsymbol{\nabla}}'\left(\frac{1}{|\vec{\boldsymbol{r}}-\vec{\boldsymbol{r}}'|}\right), \tag{32}$$

it follows that

$$\vec{\boldsymbol{\nabla}}\times\int\frac{\vec{\boldsymbol{A}}(\vec{\boldsymbol{r}}')}{|\vec{\boldsymbol{r}}-\vec{\boldsymbol{r}}'|}d^3r' = \int\vec{\boldsymbol{A}}(\vec{\boldsymbol{r}}')\times\vec{\boldsymbol{\nabla}}'\left(\frac{1}{|\vec{\boldsymbol{r}}-\vec{\boldsymbol{r}}'|}\right)d^3r',$$

To evaluate this last integral, we make use of the identity

$$\vec{\boldsymbol{\nabla}}'\times\left(\frac{\vec{\boldsymbol{A}}(\vec{\boldsymbol{r}}')}{|\vec{\boldsymbol{r}}-\vec{\boldsymbol{r}}'|}\right) = \frac{1}{|\vec{\boldsymbol{r}}-\vec{\boldsymbol{r}}'|}\vec{\boldsymbol{\nabla}}'\times\vec{\boldsymbol{A}}(\vec{\boldsymbol{r}}') - \vec{\boldsymbol{A}}(\vec{\boldsymbol{r}}')\times\vec{\boldsymbol{\nabla}}'\left(\frac{1}{|\vec{\boldsymbol{r}}-\vec{\boldsymbol{r}}'|}\right),$$

so that

$$\int\vec{\boldsymbol{A}}(\vec{\boldsymbol{r}}')\times\vec{\boldsymbol{\nabla}}'\left(\frac{1}{|\vec{\boldsymbol{r}}-\vec{\boldsymbol{r}}'|}\right)d^3r' = \int\frac{\vec{\boldsymbol{\nabla}}'\times\vec{\boldsymbol{A}}(\vec{\boldsymbol{r}}')}{|\vec{\boldsymbol{r}}-\vec{\boldsymbol{r}}'|}d^3r' - \int\vec{\boldsymbol{\nabla}}'\times\left(\frac{\vec{\boldsymbol{A}}(\vec{\boldsymbol{r}}')}{|\vec{\boldsymbol{r}}-\vec{\boldsymbol{r}}'|}\right)d^3r'.$$

We can convert the last integral to a surface integral evaluated at the surface at infinity (denoted by S_∞ below) using the analog of the divergence theorem for the curl:

$$\int\vec{\boldsymbol{\nabla}}'\times\left(\frac{\vec{\boldsymbol{A}}(\vec{\boldsymbol{r}}')}{|\vec{\boldsymbol{r}}-\vec{\boldsymbol{r}}'|}\right)d^3r' = \oint_{S_\infty}\frac{\hat{\boldsymbol{r}}'\times\vec{\boldsymbol{A}}(\vec{\boldsymbol{r}}')}{|\vec{\boldsymbol{r}}-\vec{\boldsymbol{r}}'|}r'^2 d\Omega.$$

Assuming that $\vec{\boldsymbol{A}}(\vec{\boldsymbol{r}}')$ vanishes fast enough at infinity, the surface integral vanishes. In this case,

$$\int\vec{\boldsymbol{A}}(\vec{\boldsymbol{r}}')\times\vec{\boldsymbol{\nabla}}'\left(\frac{1}{|\vec{\boldsymbol{r}}-\vec{\boldsymbol{r}}'|}\right)d^3r' = \int\frac{\vec{\boldsymbol{\nabla}}'\times\vec{\boldsymbol{A}}(\vec{\boldsymbol{r}}')}{|\vec{\boldsymbol{r}}-\vec{\boldsymbol{r}}'|}d^3r'.$$

What we have just done here is integrated by parts and dropped the surface terms, which vanish under the stated conditions. Hence,

$$-4\pi\vec{\boldsymbol{\nabla}}\times\vec{\boldsymbol{\nabla}}^{-2}(\vec{\boldsymbol{A}}) = \int\frac{\vec{\boldsymbol{\nabla}}'\times\vec{\boldsymbol{A}}(\vec{\boldsymbol{r}}')}{|\vec{\boldsymbol{r}}-\vec{\boldsymbol{r}}'|}d^3r' = -4\pi\vec{\boldsymbol{\nabla}}^{-2}(\vec{\boldsymbol{\nabla}}\times\vec{\boldsymbol{A}}),$$

after again using the definition of the inverse Laplacian given in eq. (11). Thus, we have verified eq. (29) as promised.

Second, for any scalar function $\psi(\vec{\boldsymbol{r}})$ that vanishes sufficiently fast at infinity,

$$\vec{\boldsymbol{\nabla}}\vec{\boldsymbol{\nabla}}^{-2}(\psi) = \vec{\boldsymbol{\nabla}}^{-2}(\vec{\boldsymbol{\nabla}}\psi). \tag{33}$$

This result can also be proven using eq. (32) and an integration by parts,[4]

$$-4\pi\vec{\boldsymbol{\nabla}}\vec{\boldsymbol{\nabla}}^{-2}(\psi) = \vec{\boldsymbol{\nabla}}\int\frac{\psi(\vec{\boldsymbol{r}})}{|\vec{\boldsymbol{r}}-\vec{\boldsymbol{r}}'|}d^3r' = \int\frac{\vec{\boldsymbol{\nabla}}'\psi}{|\vec{\boldsymbol{r}}-\vec{\boldsymbol{r}}'|}d^3r' = -4\pi\vec{\boldsymbol{\nabla}}^{-2}(\vec{\boldsymbol{\nabla}}\psi),$$

[4]The key step here is the identity $\vec{\boldsymbol{\nabla}}(\phi\psi) = \phi\vec{\boldsymbol{\nabla}}\psi + \psi\vec{\boldsymbol{\nabla}}\phi$, where $\phi = 1/(|\vec{\boldsymbol{r}}-\vec{\boldsymbol{r}}'|)$, and the analog of the divergence theorem for the gradient,

$$\int\vec{\boldsymbol{\nabla}}\left(\frac{\psi(\vec{\boldsymbol{r}}')}{|\vec{\boldsymbol{r}}-\vec{\boldsymbol{r}}'|}\right)d^3r' = \int_{S_\infty}\frac{\psi(\vec{\boldsymbol{r}}')\hat{\boldsymbol{r}}'}{|\vec{\boldsymbol{r}}-\vec{\boldsymbol{r}}'|}r^2 d\Omega,$$

which vanishes if $\psi(\vec{\boldsymbol{r}}')$ vanishes sufficiently fast at infinity.

which confirms eq. (33). For completeness, we note a third identity, not needed in the computations of this Appendix, which can be proven by a similar technique,

$$\vec{\boldsymbol{\nabla}}\boldsymbol{\cdot}\vec{\boldsymbol{\nabla}}^{-2}(\vec{\boldsymbol{A}}) = \vec{\boldsymbol{\nabla}}^{-2}(\vec{\boldsymbol{\nabla}}\boldsymbol{\cdot}\vec{\boldsymbol{A}})\,. \tag{34}$$

With the two identities eqs. (29) and (33) in hand, we can now verify eq. (28). First,

$$\vec{\boldsymbol{V}}_{\mathbf{tr}} = -\vec{\boldsymbol{\nabla}}\times\vec{\boldsymbol{\nabla}}^{-2}(\vec{\boldsymbol{\nabla}}\times\vec{\boldsymbol{V}}) = -\vec{\boldsymbol{\nabla}}^{-2}\left[\vec{\boldsymbol{\nabla}}\times(\vec{\boldsymbol{\nabla}}\times\vec{\boldsymbol{V}})\right] = \vec{\boldsymbol{\nabla}}^{-2}\left[\vec{\boldsymbol{\nabla}}^{2}\vec{\boldsymbol{V}} - \vec{\boldsymbol{\nabla}}(\vec{\boldsymbol{\nabla}}\boldsymbol{\cdot}\vec{\boldsymbol{V}})\right],$$

after using eq. (29) at step 2 and employing the vector identity,

$$\vec{\boldsymbol{\nabla}}\times(\vec{\boldsymbol{\nabla}}\times\vec{\boldsymbol{V}}) = \vec{\boldsymbol{\nabla}}(\vec{\boldsymbol{\nabla}}\boldsymbol{\cdot}\vec{\boldsymbol{V}}) - \vec{\boldsymbol{\nabla}}^{2}\vec{\boldsymbol{V}}\,.$$

It then follows that

$$\begin{aligned}\vec{\boldsymbol{V}}_{\mathbf{tr}} &= \vec{\boldsymbol{\nabla}}^{-2}\left[\vec{\boldsymbol{\nabla}}^{2}\vec{\boldsymbol{V}} - \vec{\boldsymbol{\nabla}}(\vec{\boldsymbol{\nabla}}\boldsymbol{\cdot}\vec{\boldsymbol{V}})\right] = \vec{\boldsymbol{V}} - \vec{\boldsymbol{\nabla}}^{-2}\left[\vec{\boldsymbol{\nabla}}(\vec{\boldsymbol{\nabla}}\boldsymbol{\cdot}\vec{\boldsymbol{V}})\right] \\ &= \vec{\boldsymbol{V}} - \vec{\boldsymbol{\nabla}}\vec{\boldsymbol{\nabla}}^{-2}(\vec{\boldsymbol{\nabla}}\boldsymbol{\cdot}\vec{\boldsymbol{V}}) = \vec{\boldsymbol{V}} - \vec{\boldsymbol{V}}_{\mathbf{long}}\,,\end{aligned} \tag{35}$$

where we have employed eq. (33) at the penultimate step. Thus, eq. (22) is confirmed.