The Poisson sum formula

The Poisson sum formula takes on a number of different forms in the literature. Here is one useful version,

$$\boxed{\frac{1}{2\pi}\sum_{n=-\infty}^{\infty} e^{inx} = \sum_{m=-\infty}^{\infty} \delta(x-2\pi m)\,.} \tag{1}$$

You will use this version of the Poisson sum formula in solving problem 14.13 of Jackson.

To prove this formula, consider the following periodic function, defined by:

$$f(x) = f(x+2\pi)\,, \quad \text{where} \quad f(x) = \frac{1}{2} - \frac{x}{2\pi} \ \text{ for } 0 \leq x \leq 2\pi. \tag{2}$$

Note that $f(x)$ is discontinuous at $x = 2\pi m$, where $m = 0, \pm 1, \pm 2, \ldots$. It follows that one can expand $f(x)$ in a Fourier series:

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx}\,,$$

where

$$c_n = \frac{1}{2\pi}\int_0^{2\pi} e^{-inx} f(x)\,dx\,. \tag{3}$$

Evaluating eq. (3) using $f(x)$ given in eq. (2), one easily obtains:

$$c_0 = 0\,, \qquad c_n = \frac{-i}{2\pi n}\,, \quad (n \neq 0)\,.$$

That is,

$$f(x) = -\frac{i}{2\pi}\sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \frac{e^{inx}}{n}\,. \tag{4}$$

Consider the derivative of $f(x)$, which we denote by $f'(x)$. Using the definition of $f(x)$ given in eq. (2), it follows that $f'(x) = -1/(2\pi)$ for $x \neq 2\pi m$ (for integer values of $m = 0, \pm 1, \pm 2, \ldots$). At $x = 2\pi m$, the discontinuity of $f(x)$ can be described by the step function $\Theta(x)$. Specifically, in the vicinity of $x = 2\pi m$,

$$f(x) = -\tfrac{1}{2} + \Theta(x - 2\pi m)\,, \qquad \text{for } x \simeq 2\pi m\,. \tag{5}$$

That is, $f(x) = -1/2$ for $x = 2\pi m - \epsilon$ and $f(x) = 1/2$ for $x = 2\pi m + \epsilon$, where $\epsilon > 0$ is an infinitesimal quantity. Taking the derivative of eq. (5) yields:

$$f'(x) = \delta(x - 2\pi m)\,, \qquad \text{for } x \simeq 2\pi m\,.$$

We conclude that:

$$f'(x) = -\frac{1}{2\pi} + \sum_{m=-\infty}^{\infty} \delta(x - 2\pi m)\,. \tag{6}$$

We can also compute $f'(x)$ by differentiating the Fourier series of $f(x)$ term-by-term. Using eq. (4), we obtain:

$$f'(x) = \frac{1}{2\pi} \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} e^{inx} = \frac{1}{2\pi}\left[-1 + \sum_{n=-\infty}^{\infty} e^{inx}\right]. \tag{7}$$

Equating eqs. (6) and (7) yields the desired result announced in eq. (1).

Actually, the most common form for the Poisson sum formula arises in the study of Fourier analysis. Given a function $f(t)$ and its Fourier transform,

$$F(\omega) \equiv \int_{-\infty}^{\infty} e^{i\omega t} f(t) dt\,, \tag{8}$$

then the Poisson sum formula is given by:[1]

$$\boxed{\sum_{m=-\infty}^{\infty} f(\alpha m) = \frac{1}{|\alpha|} \sum_{n=-\infty}^{\infty} F\left(\frac{2\pi n}{\alpha}\right), \quad \text{for real } \alpha \neq 0.} \tag{9}$$

One can derive the above result by inserting the integral expression for F on the right-hand side of eq. (9), which yields

$$\begin{aligned}
\frac{1}{|\alpha|} \sum_{n=-\infty}^{\infty} F\left(\frac{2\pi n}{\alpha}\right) &= \frac{1}{|\alpha|} \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} e^{2\pi i n t/\alpha} f(t)\, dt = \frac{1}{|\alpha|} \int_{-\infty}^{\infty} \sum_{n=-\infty}^{\infty} e^{2\pi i n t/\alpha} f(t)\, dt \\
&= \frac{2\pi}{|\alpha|} \int_{-\infty}^{\infty} f(t) \sum_{m=-\infty}^{\infty} \delta\left(\frac{2\pi t}{\alpha} - 2\pi m\right) dt \\
&= \int_{-\infty}^{\infty} f(t) \sum_{m=-\infty}^{\infty} \delta(t - \alpha m)\, dt = \sum_{m=-\infty}^{\infty} \int_{-\infty}^{\infty} f(t)\delta(t - \alpha m)\, dt \\
&= \sum_{m=-\infty}^{\infty} f(\alpha m)\,,
\end{aligned} \tag{10}$$

after employing eq. (2) and making use of the well-known identity,

$$\delta\big(\alpha(x - x')\big) = \frac{1}{|\alpha|}\delta(x - x')\,. \tag{11}$$

Thus, eq. (9) is established. For further details, see for example Refs. [1–3].

[1]Note that the corresponding formula in Ref. [1] incorrectly omits the absolute value sign that appears on the right hand side of eq. (9). The corresponding formula in Ref. [2] specifies that $\alpha > 0$, and hence no absolute value sign appears. The absolute value sign appears correctly in Ref. [3].

References

1. M.J. Lighthill, *Introduction to Fourier Analysis and Generalised Functions* (Cambridge University Press, Cambridge, UK, 1958) pp. 67–71.

2. D.S. Jones, *The Theory of Generalised Functions*, Second Edition (Cambridge University Press, Cambridge, UK, 1982) pp. 155–159.

3. Ram. P. Kanwal, *Generalized Functions—Theory and Applications*, Third Edition (Birkhäuser, Boston, 2004) pp. 168–171.