Polarization Vectors and Polarization Sums

1. Introduction

Consider a free electromagnetic field, which satisfies the source-free Maxwell equations,

$$\vec{\boldsymbol{\nabla}} \times \vec{\boldsymbol{E}} = -\frac{1}{c}\frac{\partial \vec{\boldsymbol{B}}}{\partial t}\,, \qquad \vec{\boldsymbol{\nabla}} \boldsymbol{\cdot} \vec{\boldsymbol{B}} = 0\,, \tag{1}$$

$$\vec{\boldsymbol{\nabla}} \boldsymbol{\cdot} \vec{\boldsymbol{E}} = 0\,, \qquad \vec{\boldsymbol{\nabla}} \times \vec{\boldsymbol{B}} = \frac{1}{c}\frac{\partial \vec{\boldsymbol{E}}}{\partial t}\,, \tag{2}$$

where we have employed gaussian units. In light of eq. (1), one can define a scalar potential ϕ and a vector potential $\vec{\boldsymbol{A}}$ such that,

$$\vec{\boldsymbol{E}} = -\vec{\boldsymbol{\nabla}}\phi - \frac{1}{c}\frac{\partial \vec{\boldsymbol{A}}}{\partial t}\,, \qquad \vec{\boldsymbol{B}} = \vec{\boldsymbol{\nabla}} \times \vec{\boldsymbol{A}}\,. \tag{3}$$

It is convenient to work in the Coulomb gauge where $\vec{\boldsymbol{\nabla}} \boldsymbol{\cdot} \vec{\boldsymbol{A}} = 0$. In the presence of sources ρ and $\vec{\boldsymbol{J}}$, we showed in class that ϕ and $\vec{\boldsymbol{A}}$ satisfy

$$\phi(\vec{\boldsymbol{x}}, t) = \int d^3x'\, \frac{\rho(\vec{\boldsymbol{x}}', t)}{|\vec{\boldsymbol{x}} - \vec{\boldsymbol{x}}'|}\,, \qquad \Box \vec{\boldsymbol{A}} = \frac{4\pi}{c}\vec{\boldsymbol{J}}_{tr}\,, \tag{4}$$

where $\vec{\boldsymbol{J}}_{tr} \equiv \vec{\boldsymbol{J}} - \vec{\boldsymbol{J}}_{long}$ and

$$\vec{\boldsymbol{J}}_{long} \equiv -\frac{1}{4\pi}\vec{\boldsymbol{\nabla}} \int d^3x'\, \frac{\vec{\boldsymbol{\nabla}}' \boldsymbol{\cdot} \vec{\boldsymbol{J}}(\vec{\boldsymbol{x}}', t)}{|\vec{\boldsymbol{x}} - \vec{\boldsymbol{x}}'|}\,. \tag{5}$$

Thus, when $\phi = \vec{\boldsymbol{J}} = 0$, it follows that $\phi(\vec{\boldsymbol{x}}, t) = 0$ and $\vec{\boldsymbol{A}}$ satisfies the free wave equation,

$$\Box \vec{\boldsymbol{A}}(\vec{\boldsymbol{x}}, t) = 0\,. \tag{6}$$

The most general solution to eq. (6) is

$$\vec{\boldsymbol{A}}(\vec{\boldsymbol{x}}, t) = \sum_\lambda \int \frac{d^3k}{(2\pi)^3}\left[A_0(\vec{\boldsymbol{k}}, \lambda)\, \hat{\boldsymbol{\epsilon}}_\lambda(\vec{\boldsymbol{k}})\, e^{i(\vec{\boldsymbol{k}}\boldsymbol{\cdot}\vec{\boldsymbol{x}} - \omega t)} + \text{c.c.}\right]\,, \tag{7}$$

where $A_0(\vec{\boldsymbol{k}}, \lambda)$ is a complex amplitude, the abbreviation c.c. stands for "complex conjugate" of the preceding term, and $\omega \equiv kc$ (with $k \equiv |\vec{\boldsymbol{k}}|$). Since $\phi = 0$, eq. (3) yields

$$\vec{\boldsymbol{E}}(\vec{\boldsymbol{x}}, t) = -\frac{1}{c}\frac{\partial \vec{\boldsymbol{A}}}{\partial t} = \sum_\lambda \int \frac{d^3k}{(2\pi)^3}\left[E_0(\vec{\boldsymbol{k}}, \lambda)\, \hat{\boldsymbol{\epsilon}}_\lambda(\vec{\boldsymbol{k}})\, e^{i(\vec{\boldsymbol{k}}\boldsymbol{\cdot}\vec{\boldsymbol{x}} - \omega t)} + \text{c.c.}\right]\,, \tag{8}$$

$$\vec{\boldsymbol{B}}(\vec{\boldsymbol{x}}, t) = \vec{\boldsymbol{\nabla}} \times \vec{\boldsymbol{A}} = \sum_\lambda \int \frac{d^3k}{(2\pi)^3}\left[E_0(\vec{\boldsymbol{k}}, \lambda)\, \hat{\boldsymbol{k}} \times \hat{\boldsymbol{\epsilon}}_\lambda(\vec{\boldsymbol{k}})\, e^{i(\vec{\boldsymbol{k}}\boldsymbol{\cdot}\vec{\boldsymbol{x}} - \omega t)} + \text{c.c.}\right]\,, \tag{9}$$

where $E_0(\vec{\boldsymbol{k}}, \lambda) \equiv ikA_0(\vec{\boldsymbol{k}}, \lambda)$, $\hat{\boldsymbol{\epsilon}}_\lambda(\vec{\boldsymbol{k}})$ is the *polarization vector* with two possible polarization states indicated by the subscript λ, and $\hat{\boldsymbol{k}} \equiv \vec{\boldsymbol{k}}/k$. Since the physical $\vec{\boldsymbol{E}}$ and $\vec{\boldsymbol{B}}$ fields are real vectors, it follows that the polarization vectors satisfy

$$\hat{\boldsymbol{\epsilon}}_\lambda(-\vec{\boldsymbol{k}}) = \hat{\boldsymbol{\epsilon}}^*_\lambda(\vec{\boldsymbol{k}})\,. \tag{10}$$

Note that $\vec{\boldsymbol{\nabla}}\boldsymbol{\cdot}\vec{\boldsymbol{B}} = 0$ is automatically satisfied since $\vec{\boldsymbol{k}}\boldsymbol{\cdot}\big[\hat{\boldsymbol{k}}\times\hat{\boldsymbol{\epsilon}}(\vec{\boldsymbol{k}})\big] = 0$. In contrast, the equation $\vec{\boldsymbol{\nabla}}\boldsymbol{\cdot}\vec{\boldsymbol{E}} = 0$ imposes a constraint on the polarization vectors,

$$\vec{\boldsymbol{k}}\boldsymbol{\cdot}\hat{\boldsymbol{\epsilon}}_\lambda(\vec{\boldsymbol{k}}) = 0\,, \tag{11}$$

which corresponds to the statement that electromagnetic waves are transverse. It is convenient to establish a complex basis of unit vectors that span three dimensional space: $\{\hat{\boldsymbol{\epsilon}}_1, \hat{\boldsymbol{\epsilon}}_2, \hat{\boldsymbol{k}}\}$. Although $\hat{\boldsymbol{k}}$ is a real unit vector, $\hat{\boldsymbol{\epsilon}}_1$ and $\hat{\boldsymbol{\epsilon}}_2$ can be complex unit vectors depending on the choice of basis. The complex vectors that make up the basis set are mutually orthonormal. Hence, the basis vectors satisfy:

$$\hat{\boldsymbol{\epsilon}}^*_1\boldsymbol{\cdot}\hat{\boldsymbol{\epsilon}}_1 = \hat{\boldsymbol{\epsilon}}^*_2\boldsymbol{\cdot}\hat{\boldsymbol{\epsilon}}_2 = \hat{\boldsymbol{k}}\boldsymbol{\cdot}\hat{\boldsymbol{k}} = 0\,, \tag{12}$$

$$\hat{\boldsymbol{\epsilon}}^*_\lambda\boldsymbol{\cdot}\hat{\boldsymbol{\epsilon}}_{\lambda'} = \delta_{\lambda\lambda'}\,, \qquad \text{for } \lambda,\lambda' \in \{1,2\}, \tag{13}$$

$$\vec{\boldsymbol{k}}\boldsymbol{\cdot}\hat{\boldsymbol{\epsilon}}_\lambda = \vec{\boldsymbol{k}}\boldsymbol{\cdot}\hat{\boldsymbol{\epsilon}}^*_\lambda = 0\,, \qquad \text{for } \lambda,\lambda' \in \{1,2\}, \tag{14}$$

where $\delta_{\lambda\lambda'} = 1$ if $\lambda = \lambda'$ and $\delta_{\lambda\lambda'} = 0$ if $\lambda \neq \lambda'$.

Suppose we have an electromagnetic wave moving in the z-direction, in which case $\hat{\boldsymbol{k}} = \hat{\boldsymbol{z}}$. If the wave is linearly polarized, then the polarization vectors are real: $\hat{\boldsymbol{\epsilon}}_1 = \hat{\boldsymbol{x}}$ corresponds to an x-polarized wave and $\hat{\boldsymbol{\epsilon}}_2 = \hat{\boldsymbol{y}}$ corresponds to a y-polarized wave. If the wave is circularly polarized, then the polarization vectors are complex:[1]

$$\hat{\boldsymbol{\epsilon}}_{\pm 1} \equiv \mp\frac{1}{\sqrt{2}}\big(\hat{\boldsymbol{x}} \pm i\hat{\boldsymbol{y}}\big)\,. \tag{15}$$

In the conventions employed in optics textbooks, $\hat{\boldsymbol{\epsilon}}_{+1}$ is the polarization vector of a left-circularly polarized wave (corresponding to an electric field vector that rotates in the counterclockwise direction when viewed by an observer facing the incoming wave) and $\hat{\boldsymbol{\epsilon}}_{-1}$ is the polarization vector of a right-circularly polarized wave (corresponding to an electric field vector that rotates in the clockwise direction when viewed by an observer facing the incoming wave).

When working with the circular polarization vectors, it is convenient to introduce

$$\hat{\boldsymbol{\epsilon}}_0 \equiv \hat{\boldsymbol{k}} = \hat{\boldsymbol{z}}\,. \tag{16}$$

Then, the so-called *spherical basis* vectors, which comprise the three complex unit vectors $\hat{\boldsymbol{\epsilon}}_m$ (where $m \in \{-1, 0, +1\}$), satisfy:

$$\hat{\boldsymbol{\epsilon}}^*_m\boldsymbol{\cdot}\hat{\boldsymbol{\epsilon}}_{m'} = \delta_{mm'}\,, \qquad \text{for } m \in \{-1,0,+1\}, \tag{17}$$

$$\hat{\boldsymbol{\epsilon}}^*_m = (-1)^m\,\hat{\boldsymbol{\epsilon}}_{-m}\,, \qquad \text{for } m \in \{-1,0,+1\}, \tag{18}$$

$$\hat{\boldsymbol{\epsilon}}^*_m \times \hat{\boldsymbol{\epsilon}}_{m'} = im\hat{\boldsymbol{k}}\,\delta_{mm'}\,, \qquad \text{for } m \in \{-1,+1\}, \tag{19}$$

$$i\hat{\boldsymbol{k}} \times \hat{\boldsymbol{\epsilon}}_m = m\hat{\boldsymbol{\epsilon}}_m\,, \qquad \text{for } m \in \{-1,+1\}. \tag{20}$$

[1] The choice of the ± sign in eq. (15) is conventional and has been chosen to match the phase conventions employed in the definition of the spherical harmonics [see eq. (21)]. However, note that many authors, including Jackson, omit the ± in the definition of the circularly polarized polarization vectors.

Moreover, note that the unit vectors $\hat{\boldsymbol{\epsilon}}_m$ are related to the spherical harmonics with $\ell = 1$ as follows:

$$rY_{1m}(\theta,\phi) = \sqrt{\frac{3}{4\pi}}\,\hat{\boldsymbol{\epsilon}}_m\boldsymbol{\cdot}\vec{\boldsymbol{x}}\,, \tag{21}$$

where $\vec{\boldsymbol{x}} = r\big(\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$ and $r \equiv |\vec{\boldsymbol{x}}|$ when expressed in spherical coordinates.

2. Polarization sums

In computing the angular distribution of a scattered electromagnetic wave, if the final state polarization is not observed, then one must sum over the two possible polarizations of the scattered wave. If the initial state polarization is not observed, then one must *average* over the two possible polarizations of the incoming wave. To facilitate the computation of the sum over polarizations, I will derive a very useful formula.

Recall from your quantum mechanics class that given a complete orthonormal basis $\{|n\rangle\}$, the completeness relation states that

$$\sum_n |n\rangle\,\langle n| = \mathbb{1}\,, \tag{22}$$

where $\mathbb{1}$ is the identity operator. Applying this result to the complex basis $\{\hat{\boldsymbol{\epsilon}}_1, \hat{\boldsymbol{\epsilon}}_2, \hat{\boldsymbol{k}}\}$, it follows that

$$\hat{\boldsymbol{\epsilon}}_1^*\hat{\boldsymbol{\epsilon}}_1 + \hat{\boldsymbol{\epsilon}}_2^*\hat{\boldsymbol{\epsilon}}_2 + \hat{\boldsymbol{k}}\hat{\boldsymbol{k}} = \mathbb{1}\,. \tag{23}$$

If written in terms of the vector components, eq. (23) is equivalent to

$$(\hat{\boldsymbol{\epsilon}}_1^*)_i(\hat{\boldsymbol{\epsilon}}_1)_j + (\hat{\boldsymbol{\epsilon}}_2^*)_i(\hat{\boldsymbol{\epsilon}}_2)_j + \hat{\boldsymbol{k}}_i\hat{\boldsymbol{k}}_j = \delta_{ij}\,, \tag{24}$$

where the indices $i, j \in \{1, 2, 3\}$ label the x, y and z components of the corresponding unit vector. The unit vector $\hat{\boldsymbol{k}}$ indicates the direction of the wave propagation, which is also designated by the symbol $\hat{\boldsymbol{n}}$. Hence, it follows that

$$\boxed{\sum_{\lambda=1}^{2}(\hat{\boldsymbol{\epsilon}}_\lambda)_i(\hat{\boldsymbol{\epsilon}}_\lambda^*)_j = \delta_{ij} - \hat{\boldsymbol{n}}_i\hat{\boldsymbol{n}}_j\,, \qquad \text{where } \hat{\boldsymbol{n}} \equiv \frac{\vec{\boldsymbol{k}}}{k}}\,. \tag{25}$$

Eq. (25) is called the polarization sum formula. It can be applied to any basis of polarization vectors (linear, circular, or even elliptical polarization).

3. Illustrating the use of polarization sums

To illustrate the use of the polarization sum formula, consider the angular distribution of radiation obtained via the nonrelativistic Larmour formula,

$$\frac{dP}{d\Omega} = \frac{e^2}{4\pi c^3}|\hat{\boldsymbol{n}} \times (\hat{\boldsymbol{n}} \times \vec{\boldsymbol{a}})|^2\,, \tag{26}$$

where $\vec{\boldsymbol{a}}$ is the acceleration vector of a point particle of charge e. Consider the electric field of the emitted radiation. Using the completeness relation [eq. (23)],

$$\vec{\boldsymbol{E}} = \hat{\boldsymbol{\epsilon}}_1\big(\hat{\boldsymbol{\epsilon}}_1^{*}\boldsymbol{\cdot}\vec{\boldsymbol{E}}\big) + \hat{\boldsymbol{\epsilon}}_2\big(\hat{\boldsymbol{\epsilon}}_2^{*}\boldsymbol{\cdot}\vec{\boldsymbol{E}}\big)\,, \tag{27}$$

after noting that $\hat{\boldsymbol{k}} = \hat{\boldsymbol{n}}$ and $\hat{\boldsymbol{n}}\boldsymbol{\cdot}\vec{\boldsymbol{E}} = 0$ (for a transverse wave). Hence it follows that

$$|\vec{\boldsymbol{E}}|^2 = |\hat{\boldsymbol{\epsilon}}_1^{*}\boldsymbol{\cdot}\vec{\boldsymbol{E}}|^2 + |\hat{\boldsymbol{\epsilon}}_2^{*}\boldsymbol{\cdot}\vec{\boldsymbol{E}}|^2\,. \tag{28}$$

Since Larmour's formula was derived by evaluating $|\vec{\boldsymbol{E}}|^2$ for an accelerating point charge, it follows that if the polarization vector of the emitted radiation is $\hat{\boldsymbol{\epsilon}}$ then eq. (26) must be modified as follows:

$$\frac{dP}{d\Omega} = \frac{e^2}{4\pi c^3}\big|\hat{\boldsymbol{\epsilon}}^{*}\boldsymbol{\cdot}[\hat{\boldsymbol{n}}\times(\hat{\boldsymbol{n}}\times\vec{\boldsymbol{a}})]\big|^2\,. \tag{29}$$

Using the relevant vector identities and noting that $\hat{\boldsymbol{\epsilon}}^{*}\boldsymbol{\cdot}\hat{\boldsymbol{n}} = 0$ [in light of eq. (14) after identifying $\hat{\boldsymbol{n}} = \hat{\boldsymbol{k}}$], it follows that

$$\hat{\boldsymbol{\epsilon}}^{*}\boldsymbol{\cdot}[\hat{\boldsymbol{n}}\times(\hat{\boldsymbol{n}}\times\vec{\boldsymbol{a}})] = (\hat{\boldsymbol{\epsilon}}^{*}\boldsymbol{\cdot}\hat{\boldsymbol{n}})(\vec{\boldsymbol{a}}\boldsymbol{\cdot}\hat{\boldsymbol{n}}) - (\hat{\boldsymbol{\epsilon}}^{*}\boldsymbol{\cdot}\vec{\boldsymbol{a}})(\hat{\boldsymbol{n}}\boldsymbol{\cdot}\hat{\boldsymbol{n}}) = -\hat{\boldsymbol{\epsilon}}^{*}\boldsymbol{\cdot}\vec{\boldsymbol{a}}\,. \tag{30}$$

Hence, Larmour's formula for the angular distribution for radiation observed with polarization vector $\hat{\boldsymbol{\epsilon}}$ (in the nonrelativistic limit) is given by

$$\frac{dP}{d\Omega} = \frac{e^2}{4\pi c^3}\big|\hat{\boldsymbol{\epsilon}}^{*}\boldsymbol{\cdot}\vec{\boldsymbol{a}}\big|^2\,. \tag{31}$$

In class, I showed that by using eq. (31), one can derive the famous Thomson scattering cross section formula,

$$\frac{d\sigma}{d\Omega} = \left(\frac{e^2}{mc^2}\right)^2\big|\hat{\boldsymbol{\epsilon}}_i\boldsymbol{\cdot}\hat{\boldsymbol{\epsilon}}_f^{*}\big|^2\,, \tag{32}$$

for the scattering of an electron with an incoming electromagnetic wave with polarization vector $\hat{\boldsymbol{\epsilon}}_i$ which produces an emitted electromagnetic wave with polarization vector $\hat{\boldsymbol{\epsilon}}_f$. In order to compute the scattering cross section if neither the initial polarization nor the final polarization is observed, then one must average over initial state polarizations and sum over final state polarizations. That is,

$$\left(\frac{d\sigma}{d\Omega}\right)_{\text{unpolarized}} = \frac{1}{2}\left(\frac{e^2}{mc^2}\right)^2\sum_{\lambda}\sum_{\lambda'}\big|\hat{\boldsymbol{\epsilon}}_\lambda\boldsymbol{\cdot}\hat{\boldsymbol{\epsilon}}_{\lambda'}^{*}\big|^2\,, \tag{33}$$

where we have simplified the notation by denoting the initial polarization by λ and the final polarization by λ' (thereby omitting the subscripts i and f). Assume that the incoming electromagnetic wave is propagating in the z direction and the outgoing electromagnetic wave is propagating in the $\hat{\boldsymbol{n}}$ direction. Then, eq. (25) implies that

$$\sum_{\lambda=1}^{2}(\hat{\boldsymbol{\epsilon}}_\lambda)_j(\hat{\boldsymbol{\epsilon}}_\lambda^{*})_\ell = \delta_{j\ell} - \hat{\boldsymbol{z}}_j\hat{\boldsymbol{z}}_\ell\,, \tag{34}$$

$$\sum_{\lambda'=1}^{2}(\hat{\boldsymbol{\epsilon}}_{\lambda'})_j(\hat{\boldsymbol{\epsilon}}_{\lambda'}^{*})_\ell = \delta_{j\ell} - \hat{\boldsymbol{n}}_j\hat{\boldsymbol{n}}_\ell\,. \tag{35}$$

Then eq. (38) yields

$$\frac{1}{2}\sum_{\lambda}\big|\hat{\boldsymbol{\epsilon}}_\lambda\boldsymbol{\cdot}\hat{\boldsymbol{\epsilon}}^*_{\lambda'}\big|^2=\frac{1}{2}(\hat{\boldsymbol{\epsilon}}_{\lambda'})_j(\hat{\boldsymbol{\epsilon}}^*_{\lambda'})_\ell\sum_{\lambda}(\hat{\boldsymbol{\epsilon}}_\lambda)_\ell(\hat{\boldsymbol{\epsilon}}^*_\lambda)_j=\frac{1}{2}(\hat{\boldsymbol{\epsilon}}_{\lambda'})_j(\hat{\boldsymbol{\epsilon}}^*_{\lambda'})_\ell\big(\delta_{\ell j}-\hat{\boldsymbol{z}}_\ell\hat{\boldsymbol{z}}_j\big)=\frac{1}{2}\big(1-|\hat{\boldsymbol{z}}\boldsymbol{\cdot}\hat{\boldsymbol{\epsilon}}_{\lambda'}|^2\big)\,,\tag{36}$$

where there is an implicit sum over twice-repeated indices following Einstein's summation convention. Hence it follows that

$$\begin{aligned}\sum_{\lambda}\sum_{\lambda'}\big|\hat{\boldsymbol{\epsilon}}_\lambda\boldsymbol{\cdot}\hat{\boldsymbol{\epsilon}}^*_{\lambda'}\big|^2&=\frac{1}{2}\sum_{\lambda'}\big(1-|\hat{\boldsymbol{z}}\boldsymbol{\cdot}\hat{\boldsymbol{\epsilon}}_{\lambda'}|^2\big)=1-\frac{1}{2}\sum_{\lambda'}|\hat{\boldsymbol{z}}\boldsymbol{\cdot}\hat{\boldsymbol{\epsilon}}_{\lambda'}|^2=1-\frac{1}{2}\hat{\boldsymbol{z}}_j\hat{\boldsymbol{z}}_\ell\sum_{\lambda'}(\hat{\boldsymbol{\epsilon}}_{\lambda'})_j(\hat{\boldsymbol{\epsilon}}^*_{\lambda'})_\ell\\&=1-\frac{1}{2}\hat{\boldsymbol{z}}_j\hat{\boldsymbol{z}}_\ell\big(\delta_{j\ell}-\hat{\boldsymbol{n}}_j\hat{\boldsymbol{n}}_\ell\big)=\frac{1}{2}\big[1+(\hat{\boldsymbol{z}}\boldsymbol{\cdot}\hat{\boldsymbol{n}})^2\big]=\frac{1}{2}\big(1+\cos^2\theta\big)\,,\end{aligned}\tag{37}$$

after using $\hat{\boldsymbol{n}}=(\sin\theta\cos\phi,\sin\theta\sin\phi,\cos\theta)$. Hence, we conclude that the unpolarized Thomson cross section is given by,

$$\left(\frac{d\sigma}{d\Omega}\right)_{\rm unpolarized}=\frac{1}{2}\left(\frac{e^2}{mc^2}\right)^2\big(1+\cos^2\theta\big)\,.\tag{38}$$

Integrating over angles yields the famous result for the Thomson scattering cross section,

$$\begin{aligned}\sigma_T&=\frac{1}{2}\left(\frac{e^2}{mc^2}\right)^2\int\big(1+\cos^2\theta\big)d\Omega\\&=\frac{1}{2}\left(\frac{e^2}{mc^2}\right)^2 2\pi\int_{-1}^{1}\big(1+\cos^2\theta\big)d\cos\theta\\&=\frac{8\pi}{3}\left(\frac{e^2}{mc^2}\right)^2\,.\end{aligned}\tag{39}$$

We denote the *classical radius of the electron* by

$$r_c=\frac{e^2}{mc^2}\simeq 2.82\times 10^{-13}\ {\rm cm}\,,\tag{40}$$

which would physically correspond to setting the rest energy of the electron to the electrostatic energy of a uniformly charged spherical surface of radius r_c. In terms of r_c, the Thomson scattering cross section is given by

$$\sigma_T=\frac{8\pi}{3}r_c^2\,.\tag{41}$$

Note that the concept of the classical radius of the electron does not make any sense from a quantum mechanics point of view, as quantum mechanical effects relevant for the properties of the electron cannot be ignored at distance scales shorter than the Compton wavelength of the electron, $\hbar/(mc)\simeq 2.426\times 10^{-10}$ cm, which is significantly larger than r_c.

In quantum mechanics, electromagnetic radiation can also be described as photons, in which case the Thomson scattering is interpreted as corresponding to the scattering process $\gamma+e^-\to\gamma+e^-$ (which is now called Compton scattering). In quantum electrodynamics, the total unpolarized cross section for Compton scattering can be derived in perturbation theory, and the result is given by the Klein-Nishima formula,

$$\left(\frac{d\sigma}{d\Omega}\right)_{\rm unpolarized} = \frac{1}{2}\left(\frac{e^2}{mc^2}\right)^2 y^2\left(y+\frac{1}{y}-\sin^2\theta\right)\,, \tag{42}$$

where

$$y \equiv \frac{1}{1+z(1-\cos\theta)} \quad \text{and} \quad z \equiv \frac{\hbar\omega}{mc^2}\,. \tag{43}$$

Integrating over angles to obtain the total cross section, we obtain

$$\sigma_{KN} = \frac{\pi}{z^3}\left(\frac{e^2}{mc^2}\right)^2\left[\frac{2z(2+8z+9z^2+z^3)}{(1+2z)^2} - (2+2z-z^2)\ln(1+2z)\right]\,. \tag{44}$$

In the classical limit of long wavelengths (corresponding to $\hbar\omega \ll mc^2$), $z \to 0$ (and $y \to 1$) and the Klein-Nishima cross section tends to the Thomson cross section obtained in eqs. (38) and (39). One can check explicitly that eq. (44) reduces to eq. (39) in the limit as $z \to 0$ by expanding out the logarithm in eq. (44) for small z and showing that the term within the brackets in eq. (44) behaves as $8z^3/3$ as $z \to 0$. In case you are curious, Mathematica yields

$$\frac{2z(2+8z+9z^2+z^3)}{(1+2z)^2} - (2+2z-z^2)\ln(1+2z) = \frac{8z^3}{3}\left[1-2z+\mathcal{O}(z^2)\right]\,. \tag{45}$$