The radial Green function

In eqs. (6.36)–(6.40), Jackson derives an expression for the Green function, $G_k(\vec{\boldsymbol{x}},\vec{\boldsymbol{x}}')$ that satisfies,

$$(\vec{\boldsymbol{\nabla}}^2+k^2)G_k(\vec{\boldsymbol{x}},\vec{\boldsymbol{x}}')=-\delta^3(\vec{\boldsymbol{x}}-\vec{\boldsymbol{x}}')\,, \tag{1}$$

where I have omitted an overall factor of 4π following eqs. (9.93)–(9.95) of Jackson. After imposing the boundary condition corresponding to *outgoing* waves, Jackson obtains,[1]

$$G_k(\vec{\boldsymbol{x}},\vec{\boldsymbol{x}}')=\frac{e^{ik|\vec{\boldsymbol{x}}-\vec{\boldsymbol{x}}'|}}{4\pi|\vec{\boldsymbol{x}}-\vec{\boldsymbol{x}}'|}\,. \tag{2}$$

We can expand this expression in spherical harmonics,

$$\frac{e^{ik|\vec{\boldsymbol{x}}-\vec{\boldsymbol{x}}'|}}{4\pi|\vec{\boldsymbol{x}}-\vec{\boldsymbol{x}}'|}=\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}g_\ell(r,r')Y^*_{\ell m}(\Omega')Y_{\ell m}(\Omega)\,, \tag{3}$$

where $r\equiv|\vec{\boldsymbol{x}}|$, $r'\equiv|\vec{\boldsymbol{x}}'|$, $\Omega=(\theta,\phi)$ and $\Omega'=(\theta',\phi')$, where θ and ϕ [θ' and ϕ'] are the polar and azimuthal angles of the vector $\vec{\boldsymbol{x}}$ [$\vec{\boldsymbol{x}}'$] with respect to a fixed coordinate system. In this note, I will derive an expression for the radial Green function, $g_\ell(r,r')$.[2] The $k\to 0$ limit of eq. (3), which is given by eq. (3.70) of Jackson will serve as a check of our result (see Appendix B).

The first observation is that the left hand side of eq. (3) is invariant under the interchange of $\vec{\boldsymbol{x}}\leftrightarrow\vec{\boldsymbol{x}}'$. Hence,

$$\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}g_\ell(r,r')Y^*_{\ell m}(\Omega')Y_{\ell m}(\Omega)=\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}g_\ell(r',r)Y^*_{\ell m}(\Omega)Y_{\ell m}(\Omega')\,. \tag{4}$$

However, recall the addition theorem for spherical harmonics [Jackson eq. (3.62)],

$$P_\ell(\cos\gamma)=\frac{4\pi}{2\ell+1}\sum_{m=-\ell}^{\ell}Y^*_{\ell m}(\Omega')Y_{\ell m}(\Omega)\,, \tag{5}$$

where γ is the angle between $\vec{\boldsymbol{x}}$ and $\vec{\boldsymbol{x}}'$. Since $P_\ell(\cos\gamma)$ is a real function, it follows that

$$\sum_{m=-\ell}^{\ell}Y^*_{\ell m}(\Omega')Y_{\ell m}(\Omega)=\sum_{m=-\ell}^{\ell}Y_{\ell m}(\Omega')Y^*_{\ell m}(\Omega)\,. \tag{6}$$

Consequently, eqs. (4) and (6) yield,

$$\sum_{\ell=0}^{\infty}\big[g_\ell(r,r')-g_\ell(r',r)\big]\sum_{m=-\ell}^{\ell}Y^*_{\ell m}(\Omega')Y_{\ell m}(\Omega)=0\,. \tag{7}$$

[1] For completeness, a derivation of eq. (2) is given in Appendix A of these notes.

[2] A better notation for the radial Green function would be $g_k^{(\ell)}(r,r')$ to emphasize the dependence on k. However, I will stick with the notation that Jackson uses.

After employing eq. (5) and noting that the Legendre polynomials constitute a linearly independent set of functions, it follows that

$$g_\ell(r,r') = g_\ell(r',r)\,. \tag{8}$$

The second observation is that we can obtain a differential equation for $g_\ell(r,r')$ by inserting eq. (3) into eq. (1). Recall that in spherical coordinates,

$$\vec{\boldsymbol{\nabla}}^2 = \frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} - \frac{\vec{\boldsymbol{L}}^2}{r^2}\,, \tag{9}$$

where $\vec{\boldsymbol{L}}^2$ is a differential operator with the property that $\vec{\boldsymbol{L}}^2\, Y_{\ell m}(\Omega) = \ell(\ell+1)Y_{\ell m}(\Omega)$. Hence, it follows that

$$\begin{aligned}(\vec{\boldsymbol{\nabla}}^2 + k^2)g_\ell(r,r')Y_{\ell m}(\Omega) &= \left(\frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} + k^2 - \frac{\vec{\boldsymbol{L}}^2}{r^2}\right) g_\ell(r,r')Y_{\ell m}(\Omega) \\ &= \left(\frac{d^2}{dr^2} + \frac{2}{r}\frac{d}{dr} + k^2 - \frac{\ell(\ell+1)}{r^2}\right) g_\ell(r,r')Y_{\ell m}(\Omega)\,. \end{aligned} \tag{10}$$

It follows that $g_\ell(r,r')$ must satisfy the equation,

$$\boxed{\left(\frac{d^2}{dr^2} + \frac{2}{r}\frac{d}{dr} + k^2 - \frac{\ell(\ell+1)}{r^2}\right) g_\ell(r,r') = -\frac{1}{r^2}\delta(r-r')} \tag{11}$$

which defines the radial Green function. To show that eq. (11) is equivalent to eqs. (1)–(3),

$$\begin{aligned}(\vec{\boldsymbol{\nabla}}^2 + k^2)G_k(\vec{\boldsymbol{x}},\vec{\boldsymbol{x}}') &= (\vec{\boldsymbol{\nabla}}^2 + k^2)\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell} g_\ell(r,r')Y^*_{\ell m}(\Omega')Y_{\ell m}(\Omega) \\ &= -\frac{1}{r^2}\delta(r-r')\sum_{m=-\ell}^{\ell} Y^*_{\ell m}(\Omega')Y_{\ell m}(\Omega) = -\frac{1}{r^2}\delta(r-r')\delta(\Omega-\Omega') \\ &= -\delta^3(\vec{\boldsymbol{x}}-\vec{\boldsymbol{x}}')\,, \end{aligned} \tag{12}$$

after using the completeness relation of the spherical harmonics [cf. eq. (3.56) of Jackson]:[3]

$$\sum_{m=-\ell}^{\ell} Y^*_{\ell m}(\Omega')Y_{\ell m}(\Omega) = \delta(\Omega-\Omega')\,, \tag{13}$$

and the expression for the three dimensional delta function in spherical coordinates.

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[3] To derive eq. (13), consider the expansion of an arbitrary function of the angles $\Omega = (\theta,\phi)$ in terms of the spherical harmonics, $f(\Omega) = \sum_{\ell,m} c_{\ell m}Y_{\ell m}(\Omega)$. To obtain the coefficients $c_{\ell m}$, we employ the orthonormality relation [eq. (3.55) of Jackson]:

$$\int Y^*_{\ell' m'}(\Omega)Y_{\ell m}(\Omega)\,d\Omega = \delta_{\ell\ell'}\delta_{mm'}\,.$$

It then follows that

$$c_{\ell m} = \int f(\Omega')Y^*_{\ell' m'}(\Omega')d\Omega'\,.$$

Plugging this result back into the expansion of $f(\Omega)$, one obtains an identity if eq. (13) is satisfied.

We now proceed to solve the differential equation [eq. (11)] for $g_\ell(r,r')$. First we consider the case of $r\neq r'$. Then,

$$\left(\frac{d^2}{dr^2}+\frac{2}{r}\,\frac{d}{dr}+k^2-\frac{\ell(\ell+1)}{r^2}\right)g_\ell(r,r')=0\,,\quad \text{for } r\neq r', \tag{14}$$

which we recognize as the equation for spherical Bessel functions. In the analysis that follows, we will need to know the small argument and large argument behaviors of the spherical Bessel functions. In the limit as $x\to 0$,

$$j_\ell(x)=\frac{x^\ell}{(2\ell+1)!!}\big[1+\mathcal{O}(x)\big]\,,\qquad n_\ell(x)=-\frac{(2\ell-1)!!}{x^{\ell+1}}\big[1+\mathcal{O}(x)\big]\,, \tag{15}$$

where $(2\ell+1)!!=(2\ell+1)(2\ell-1)\cdots 5{\cdot}3{\cdot}1$ [for nonnegative integers ℓ] and $(-1)!!=1$. In the limit of $x\to\infty$,

$$j_\ell(x)=\frac{1}{x}\sin\left(x-\tfrac{1}{2}\ell\pi\right)\left[1+\mathcal{O}\left(\frac{1}{x}\right)\right],\qquad n_\ell(x)=-\frac{1}{x}\cos\left(x-\tfrac{1}{2}\ell\pi\right)\left[1+\mathcal{O}\left(\frac{1}{x}\right)\right]. \tag{16}$$

The spherical Hankel functions are defined by

$$h_\ell^{(1)}(x)\equiv j_\ell(x)+in_\ell(x)\,,\qquad h_\ell^{(2)}(x)\equiv j_\ell(x)-in_\ell(x)=\big[h_\ell^{(1)}(x)\big]^*. \tag{17}$$

Hence, it follows that as $x\to\infty$,

$$h_\ell^{(1)}(x)=(-i)^{\ell+1}\frac{e^{ix}}{x}\left[1+\mathcal{O}\left(\frac{1}{x}\right)\right],\qquad h_\ell^{(2)}(x)=i^{\ell+1}\frac{e^{-ix}}{x}\left[1+\mathcal{O}\left(\frac{1}{x}\right)\right]. \tag{18}$$

We shall solve eq. (14) by treating the cases of $r<r'$ and $r>r'$ separately. First, if $r<r'$ then $g_\ell(r,r')=A(r')j_\ell(kr)$, where we reject the solution proportional to $n_\ell(kr)$ under the requirement that a physical solution must be nonsingular as $r\to 0$. Second, if $r>r'$, we shall impose the condition that the solution behave as an outgoing spherical wave as $r\to\infty$. Hence, we conclude that if $r>r'$ then $g_\ell(r,r')=B(r')h_\ell^{(1)}(kr)$, since the solution proportional to $h^{(2)}(kr)$ behaves like an incoming spherical wave as $r\to\infty$ [cf. eq. (18)]. Combining these two results using eq. (8), which asserts that $g_\ell(r,r')$ is symmetric under the interchange of $r\to r'$, we can conclude that

$$g_\ell(r,r')=Cj_\ell(kr_<)h_\ell^{(1)}(kr_>)\,,\quad \text{for } r\neq r', \tag{19}$$

where $r_<\equiv\min\{r,r'\}$ and $r_>\equiv\max\{r,r'\}$. The constant C is independent of r and r' and can be determined by integrating eq. (11) from $r=r'-\epsilon$ to $r=r'+\epsilon$, where ϵ is a positive infinitesimal quantity. It then follows that

$$\left.\frac{\partial g_\ell(r,r')}{\partial r}\right|_{r=r'+\epsilon}-\left.\frac{\partial g_\ell(r,r')}{\partial r}\right|_{r=r'-\epsilon}+\frac{2}{r'}\left[g_\ell(r'-\epsilon,r')-g_\ell(r'+\epsilon,r')\right]$$
$$+\int_{r'-\epsilon}^{r'+\epsilon}\left(k^2-\frac{\ell(\ell+1)}{r^2}\right)g_\ell(r,r')dr=-\frac{1}{r'^2}\,. \tag{20}$$

In light of eq. (19), $g_\ell(r,r')$ is continuous at $r=r'$. Thus as $\epsilon\to 0$, eq. (20) reduces to,

$$\left.\frac{\partial g_\ell(r,r')}{\partial r}\right|_{r=r'+\epsilon} - \left.\frac{\partial g_\ell(r,r')}{\partial r}\right|_{r=r'-\epsilon} = -\frac{1}{r'^2}\,. \tag{21}$$

Plugging in eq. (19) and taking the limit as $\epsilon\to 0$ yields an equation for the constant C,

$$C\left[j_\ell(kr)\left(\frac{dh_\ell^{(1)}(kr)}{dr}\right)_{r=r'} - h_\ell^{(1)}(kr')\left(\frac{j_\ell(kr)}{dr}\right)_{r=r'}\right] = -\frac{1}{r'^2}\,. \tag{22}$$

Eq. (22) is an identity that must hold for all values of r' and k. Consequently, the easiest way to evaluate C is to consider the small kr' behavior of eq. (22). In this case, we can employ the small argument expressions given in eq. (16) and the definition of the spherical Hankel function [eq. (17)] to obtain,

$$j_\ell(kr) = \frac{(kr)^\ell}{(2\ell+1)!!}\left[1+\mathcal{O}(kr)\right]\,, \qquad h_\ell^{(1)}(kr) = \frac{-i(2\ell-1)!!}{(kr)^{\ell+1}}\left[1+\mathcal{O}(kr)\right]. \tag{23}$$

Hence, eq. (22) yields,

$$\begin{aligned} -\frac{1}{Cr^2} &= \frac{(kr)^\ell}{(2\ell+1)!!}\,k\,\frac{(-i)(2\ell-1)!!}{(kr)^{\ell+2}}(-\ell-1) - \frac{k\ell(kr)^{\ell-1}}{(2\ell+1)!!}\,k\,\frac{(-i)(2\ell-1)!!}{(kr)^{\ell+1}} \\ &= \frac{i}{k}(2\ell+1)\frac{(2\ell-1)!!}{(2\ell+1)!!}\,\frac{1}{r^2} = \frac{i}{kr^2}\,. \end{aligned} \tag{24}$$

That is, $C=ik$. Thus, the radial Green function is,

$$\boxed{g_\ell(r,r') = ikj_\ell(kr_<)h_\ell^{(1)}(kr_>)\,.} \tag{25}$$

Another Derivation of the radial Green function

A slightly fancier technique for deriving eq. (25) starts with the completeness relation of the spherical Bessel functions,

$$\int_0^\infty j_\ell(kr)j_\ell(kr')k^2\,dk = \frac{\pi}{2r^2}\delta(r-r')\,, \tag{26}$$

where $j_\ell(kr)$ satisfies,

$$\left(\frac{d^2}{dr^2}+\frac{2}{r}\frac{d}{dr}-\frac{\ell(\ell+1)}{r^2}\right)j_\ell(kr) = -k^2 j_\ell(kr)\,. \tag{27}$$

This suggests that one can solve eq. (11) by writing

$$g_\ell(r,r') = \int_0^\infty j_\ell(k'r)R_\ell(k',r')k'^2\,dk'\,. \tag{28}$$

Plugging the above expression for $g_\ell(r,r')$ into eq. (11) and making use of eq. (26) yields,

$$\int_0^\infty (k^2-k'^2)j_\ell(k'r)R_\ell(k',r')k'^2\,dk' = -\frac{2}{\pi}\int_0^\infty j_\ell(k'r)j_\ell(k'r')k'^2\,dk'\,. \tag{29}$$

Hence, it follows that $R_\ell(k',r')$ solves an algebraic equation,

$$(k^2 - k'^2)R_\ell(k',r') = -\frac{2}{\pi} j_\ell(k'r')\,. \tag{30}$$

Solving for $R_\ell(k',r')$ yields,

$$R_\ell(k',r')k'^2 = -\frac{2j_\ell(k'r')}{\pi(k^2-k'^2)}\,. \tag{31}$$

Inserting this result back into eq. (28), we end up with:

$$g_\ell(r,r') = -\frac{2}{\pi}\int_0^\infty \frac{j_\ell(k'r)j_\ell(k'r')}{k^2-k'^2}\,k'^2\,dk'\,. \tag{32}$$

Noting that

$$j_\ell(-x) = (-1)^\ell j_\ell(x)\,, \tag{33}$$

we see that the integrand in eq. (32) is an even function of k'. Hence, an equivalent form for eq. (32) is

$$g_\ell(r,r') = -\frac{1}{\pi}\int_{-\infty}^\infty \frac{j_\ell(k'r)j_\ell(k'r')}{k^2-k'^2}\,k'^2\,dk'\,. \tag{34}$$

Unfortunately, eq. (34) is not well defined due to the singularities at $k' = \pm k$ along the path of integration.

However, as in Appendix A, one can deform the contour around the singular points, or equivalently we can give k an infinitesimal imaginary part. The choice of the deformation depends on the desired boundary conditions for the problem. As noted earlier, we require that $g(r,r')$ should be nonsingular as $r_< \to 0$ and should behave as an outgoing spherical wave as $r_> \to \infty$. These requirements uniquely specify the required deformation of the contour. As we shall demonstrate below by an explicit computation, the deformation that yields the correct boundary conditions is,[4]

$$g_\ell(r,r') = -\frac{1}{\pi}\int_{-\infty}^\infty \frac{j_\ell(k'r)j_\ell(k'r')}{k^2-k'^2+i\varepsilon}\,k'^2\,dk'\,, \tag{35}$$

where ε is a positive infinitesimal quantity that will be taken to zero at the end of the computation. Since $j_\ell(kr)$ is analytic in the complex k' plane, the only singularities of the integrand occur when $k'^2 = k^2 + i\varepsilon$. That is, the integrand possesses two poles at $k' = k + i\varepsilon$ and $k' = -k - i\varepsilon$ (after absorbing a factor of 2 in the infinitesimal quantity ε). Note that although k' is integrated over the entire real axis in eq. (35), the quantity $k \equiv \omega/c$ remains a real positive quantity.

[4]Eq. (35) is also obtained in Charles J. Joachain, *Quantum Collision Theory* (North-Holland Publishing Company, Amsterdam, The Netherlands, 1975) pp. 122–123 [although my derivation of eq. (35) is more direct]. The subsequent analysis presented below follows the same steps presented by Joachain. Note that the radial Green function in Joachain's book is defined with the opposite sign to the one employed in these notes since Joachain omits the minus sign in eq. (1).

To perform the integral exhibited in eq. (35), we shall use the following relation,

$$j_\ell(x) = \tfrac{1}{2}\big[h_\ell^{(1)}(x) + h_\ell^{(2)}(x)\big] = \tfrac{1}{2}\big[h_\ell^{(1)}(x) + (-1)^\ell h_\ell^{(1)}(-x)\big]\,. \tag{36}$$

Then, if $r < r'$, the we can write

$$g_\ell(r,r') = \frac{1}{2\pi}\left\{\int_{-\infty}^{\infty} \frac{j_\ell(k'r)h_\ell^{(1)}(k'r')\,k'^2\,dk'}{(k'-k-i\varepsilon)(k'+k+i\varepsilon)} + (-1)^\ell \int_{-\infty}^{\infty} \frac{j_\ell(k'r)h_\ell^{(1)}(-k'r')\,k'^2\,dk'}{(k'-k-i\varepsilon)(k'+k-+\varepsilon)}\right\}. \tag{37}$$

We can now extend the integration contour to the complex k'-plane. Because $r < r'$ it follows from eqs. (18) and (36) that $j_\ell(k'r)h_\ell^{(1)}(k'r')$ is exponentially damped in the upper half complex k'-plane and $j_\ell(k'r)h_\ell^{(1)}(-k'r')$ is exponentially damped in the lower half complex k'-plane as $|k'| \to \infty$. Thus, in the first integral in eq. (37), we may close the contour in the upper half complex k'-plane. The resulting closed contour C is counterclockwise and contains only one pole inside C at $k' = k + i\varepsilon$. Hence, we may use the residue theorem of complex analysis to obtain

$$\oint_C \frac{j_\ell(k'r)h_\ell^{(1)}(k'r')\,k'^2\,dk'}{(k'-k-i\varepsilon)(k'+k+i\varepsilon)} = \pi i k j_\ell(kr)h_\ell^{(1)}(kr')\,. \tag{38}$$

Likewise, in the second integral in eq. (37), we may close the contour in the lower half complex k'-plane. The resulting closed contour C' is clockwise and contains only one pole inside C' at $k' = -k - i\varepsilon$. Hence, we may use the residue theorem of complex analysis to obtain

$$\oint_{C'} \frac{j_\ell(k'r)h_\ell^{(1)}(-k'r')\,k'^2\,dk'}{(k'-k-i\varepsilon)(k'+k+i\varepsilon)} = \pi i k j_\ell(-kr)h_\ell^{(1)}(kr')\,, \tag{39}$$

after remembering to include the extra minus sign due to the clockwise contour. Finally, after employing eq. (33) and noting that $(-1)^{2\ell} = 1$, we see that eq. (37) yields,

$$g_\ell(r,r') = ik j_\ell(kr)h_\ell^{(1)}(kr')\,, \quad \text{for } r < r'. \tag{40}$$

If $r > r'$, then we can write,

$$g_\ell(r,r') = \frac{1}{2\pi}\left\{\int_{-\infty}^{\infty} \frac{h_\ell^{(1)}(k'r)j_\ell(k'r')\,k'^2\,dk'}{(k'-k-i\varepsilon)(k'+k+i\varepsilon)} + (-1)^\ell \int_{-\infty}^{\infty} \frac{h_\ell^{(1)}(-k'r)j_\ell(k'r')\,k'^2\,dk'}{(k'-k-i\varepsilon)(k'+k-+\varepsilon)}\right\}. \tag{41}$$

Because $r > r'$, it follows from eqs. (18) and (36) that $h_\ell^{(1)}(k'r)j_\ell(k'r')$ is exponentially damped in the upper half complex k'-plane and $h_\ell^{(1)}(-k'r)j_\ell(k'r')$ is exponentially damped in the lower half complex k'-plane as $|k'| \to \infty$. A similar analysis as above then yields,

$$g_\ell(r,r') = ik h_\ell^{(1)}(kr)j_\ell(kr')\,, \quad \text{for } r > r'. \tag{42}$$

We can combine the results of eqs. (40) and (42) to obtain,

$$g_\ell(r,r') = ik j_\ell(kr_<)h_\ell^{(1)}(kr_>)\,, \quad \text{where } r_< \equiv \min\{r,r'\} \text{ and } r_> \equiv \max\{r,r'\}. \tag{43}$$

Thus, we have successfully reproduced eq. (25).

APPENDIX A: The Green function of the Helmholtz equation

In this Appendix, we shall provide an explicit derivation of eq. (2), which is a solution to eq. (1) subject to the boundary condition of outgoing waves. Using translational invariance, it follows that $G_k(\vec{\boldsymbol{x}},\vec{\boldsymbol{x}}\,') = G_k(\vec{\boldsymbol{x}}-\vec{\boldsymbol{x}}\,')$. We can turn the differential equation [eq. (1)] into an algebraic equation by employing the Fourier transform,

$$G_k(\vec{\boldsymbol{x}}) = \frac{1}{(2\pi)^3}\int d^3q\,\widetilde{G}(\vec{\boldsymbol{q}})\,e^{i\vec{\boldsymbol{q}}\cdot\vec{\boldsymbol{x}}}\,, \tag{44}$$

Acting on both sides of this equation with $(\vec{\boldsymbol{\nabla}}^2 + k^2)$ and making use of the integral representation of the delta function,

$$\delta^3(\vec{\boldsymbol{x}}) = \frac{1}{(2\pi)^3}\int d^3q\,e^{i\vec{\boldsymbol{q}}\cdot\vec{\boldsymbol{x}}}\,, \tag{45}$$

we end up with the algebraic equation, $(k^2-q^2)\widetilde{G}(\vec{\boldsymbol{q}}) = -1$. Hence,

$$\widetilde{G}(\vec{\boldsymbol{q}}) = \frac{1}{q^2-k^2}\,, \tag{46}$$

where $q \equiv |\vec{\boldsymbol{q}}|$. Plugging this result back into eq. (44) yields

$$G_k(\vec{\boldsymbol{x}}) = \frac{1}{(2\pi)^3}\int d^3q\,\frac{e^{i\vec{\boldsymbol{q}}\cdot\vec{\boldsymbol{x}}}}{q^2-k^2}\,. \tag{47}$$

Writing $d^3q = q^2\,dq\,d\Omega = q^2\,dq\,d\cos\theta\,d\phi = 2\pi q^2\,dq\,d\cos\theta$ (one can freely integrate over ϕ since there is no ϕ dependence in the integrand above) and $\vec{\boldsymbol{q}}\cdot\vec{\boldsymbol{x}} = qr\cos\theta$ where $r \equiv |\vec{\boldsymbol{x}}|$ and θ is the angle between $\vec{\boldsymbol{q}}$ and $\vec{\boldsymbol{x}}$, it follows that

$$\begin{aligned} G_k(\vec{\boldsymbol{x}}) &= \frac{1}{(2\pi)^2}\int_0^\infty \frac{q^2\,dq}{q^2-k^2}\int_{-1}^{1} e^{iqr\cos\theta} = \frac{1}{4\pi^2}\int_0^\infty \frac{q^2\,dq}{q^2-k^2}\,\frac{1}{iqr}\big(e^{iqr}-e^{-iqr}\big) \\ &= \frac{1}{2\pi^2 r}\int_0^\infty \frac{q\sin(qr)\,dq}{q^2-k^2} = \frac{1}{4\pi^2 r}\int_{-\infty}^\infty \frac{q\sin(qr)\,dq}{q^2-k^2}\,, \end{aligned} \tag{48}$$

where we used the fact that the integrand above is an even function of q to extend the limits of integration from $(0,\infty)$ to $(-\infty,\infty)$. The integral in eq. (48) is undefined due to the singularity at $q = \pm k$ along the path of integration. However, we can implement the boundary condition corresponding to outgoing waves by deforming the contour or equivalently by adding an infinitesimal imaginary part to k as follows,

$$G_k(\vec{\boldsymbol{x}}) = \frac{1}{4\pi^2 r}\int_{-\infty}^\infty \frac{q\sin(qr)\,dq}{q^2-k^2-i\varepsilon}\,, \tag{49}$$

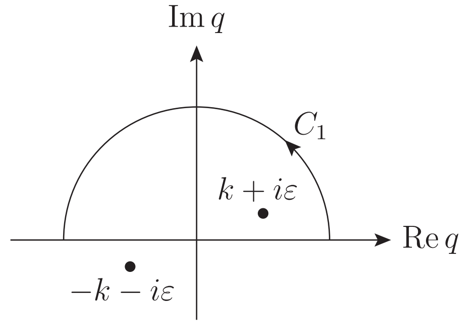
where ε is a positive infinitesimal quantity that will be taken to zero at the end of the calculation. Factoring the denominator yields,

$$q^2-k^2-i\varepsilon = (q-k-i\varepsilon)(q+k+i\varepsilon)\,, \tag{50}$$

after absorbing a factor of two into the definition of ε on the right hand side above. Note that $k=\omega/c$ is a real positive quantity.

To perform the integral given in eq. (49), we first write $2i \sin qr = e^{iqr} - e^{-iqr}$. Consider first,

$$I_1(k,\varepsilon) \equiv \int_{-\infty}^{\infty} \frac{q e^{iqr}\, dq}{(q-k-i\varepsilon)(q+k+i\varepsilon)}$$



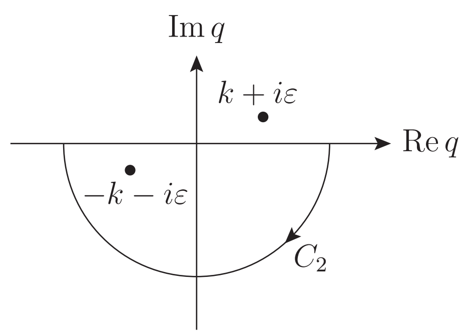
where C_1 is the closed contour shown above, and the radius of the contour is taken to infinity. Note that the integrand is exponentially damped along the semicircular part of the contour C_1 and thus the contribution to the integral along the semicircular arc goes to zero as the radius of the semicircle is taken to infinity. Inside the counterclockwise contour C_1 there exists a simple pole at $q = k + i\varepsilon$ (since by assumption, $\varepsilon > 0$). Thus, by the residue theorem of complex analysis,

$$\lim_{\varepsilon\to 0} I_1(k,\varepsilon) = 2\pi i \,\text{Res}\left(\frac{q e^{iqr}}{q^2-k^2-i\varepsilon}\right) = \pi i e^{ikr}\,, \tag{51}$$

where $\text{Res} f(q) = \lim_{q\to q_0}(q-q_0)f(q)$ is the residue due to a simple pole at $q = q_0$.

Next, we consider

$$I_2(k,\varepsilon) \equiv \int_{-\infty}^{\infty} \frac{q e^{-iqr}\, dq}{(q-k-i\varepsilon)(q+k+i\varepsilon)}$$



where the contour C_2 is now closed in the lower half plane. The integrand is exponentially damped along the semicircular part of the contour C_2 and thus the contribution to the integral along the semicircular arc goes to zero as the radius of the semicircle is taken to infinity. Inside the clockwise contour C_2 there exists a simple pole at $q = -k - i\varepsilon$. Thus, by the residue theorem of complex analysis,

$$\lim_{\varepsilon\to 0} I_2(k,\varepsilon) = -2\pi i \,\text{Res}\left(\frac{q e^{-iqr}}{q^2-k^2-i\varepsilon}\right) = -\pi i e^{ikr}\,, \tag{52}$$

where the extra minus sign is due to the clockwise orientation of C_2.

Using eqs. (51) and (52), it follows that

$$G_k(\vec{\boldsymbol{x}}) = \frac{1}{4\pi^2 r}\int_{-\infty}^{\infty} \frac{q\sin(qr)\, dq}{q^2-k^2-i\varepsilon} = \frac{1}{8i\pi^2 r}\lim_{\varepsilon\to 0}\big[I_1(k,\varepsilon) - I_2(k,\varepsilon)\big] = \frac{e^{ikr}}{4\pi r}\,. \tag{53}$$

Noting that $r \equiv |\vec{\boldsymbol{x}}|$, we conclude that

$$G_k(\vec{\boldsymbol{x}}, \vec{\boldsymbol{x}}') = G_k(\vec{\boldsymbol{x}} - \vec{\boldsymbol{x}}') = \frac{e^{ik|\vec{\boldsymbol{x}} - \vec{\boldsymbol{x}}'|}}{4\pi|\vec{\boldsymbol{x}} - \vec{\boldsymbol{x}}'|}\,. \tag{54}$$

which confirms eq. (2).

<u>*REMARKS:*</u>

Eq. (49) provides an integral representation of $G_k(\vec{\boldsymbol{x}}, \vec{\boldsymbol{x}}')$,

$$G_k(\vec{\boldsymbol{x}}, \vec{\boldsymbol{x}}') = \frac{1}{4\pi^2|\vec{\boldsymbol{x}} - \vec{\boldsymbol{x}}'|}\int_{-\infty}^{\infty} \frac{\sin\bigl(q|\vec{\boldsymbol{x}} - \vec{\boldsymbol{x}}'|\bigr)}{q^2 - k^2 - i\varepsilon}\, q\, dq\,. \tag{55}$$

If we expand in terms of spherical harmonics, then [cf. eq. (3)],

$$G_k(\vec{\boldsymbol{x}}, \vec{\boldsymbol{x}}') = \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell} g_\ell(r, r')Y^*_{\ell m}(\Omega')Y_{\ell m}(\Omega)\,, \tag{56}$$

where the radial Green function is given by an integral representation exhibited in eq. (35), which we rewrite below,

$$g_\ell(r, r') = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{j_\ell(qr)j_\ell(qr')}{q^2 - k^2 - i\varepsilon}\, q^2\, dq\,. \tag{57}$$

As a result, we obtain the following interesting expansion,

$$\frac{\sin\bigl(q|\vec{\boldsymbol{x}} - \vec{\boldsymbol{x}}'|\bigr)}{4\pi q|\vec{\boldsymbol{x}} - \vec{\boldsymbol{x}}'|} = \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell} j_\ell(qr)j_\ell(qr')Y^*_{\ell m}(\Omega')Y_{\ell m}(\Omega)\,. \tag{58}$$

Using the addition theorem for spherical harmonics given in eq. (5), it then follows that

$$\frac{\sin\bigl(q|\vec{\boldsymbol{x}} - \vec{\boldsymbol{x}}'|\bigr)}{q|\vec{\boldsymbol{x}} - \vec{\boldsymbol{x}}'|} = \sum_{\ell=0}^{\infty}(2\ell+1)j_\ell(qr)j_\ell(qr')P_\ell(\cos\theta)\,, \tag{59}$$

where θ is the angle between the vectors $\vec{\boldsymbol{x}}$ and $\vec{\boldsymbol{x}}'$. As a sanity check, if one sets $\vec{\boldsymbol{x}}' = 0$ and uses $j_\ell(0) = \delta_{\ell 0}$ and $P_0(\cos\theta) = 1$, then eq. (59) yields $j_0(qr) = \sin(qr)/(qr)$, which is correct.

Finally, by employing the orthogonality relation satisfied by the Legendre polynomials,

$$\int_{-1}^{2} P_\ell(\cos\theta)P_{\ell'}(\cos\theta)\, d\cos\theta = \frac{2}{2\ell+1}\,\delta_{\ell\ell'}\,, \tag{60}$$

we can derive an interesting integral representation for the product of two spherical Bessel functions,

$$j_\ell(qr)j_\ell(qr') = \frac{1}{2}\int_{-1}^{1} \frac{\sin\Bigl(q\sqrt{r^2 + r'^2 - 2rr'\cos\theta}\Bigr)}{q\sqrt{r^2 + r'^2 - 2rr'\cos\theta}}\, P_\ell(\cos\theta)\, d\cos\theta\,. \tag{61}$$

APPENDIX B: The $\boldsymbol{k = 0}$ limit

Having found an explicit formula for the radial Green function, one can now plug eq. (25) into eq. (3) to obtain,

$$\boxed{\frac{e^{ik|\vec{x}-\vec{x}\,'|}}{4\pi|\vec{x}-\vec{x}\,'|} = ik\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell} j_\ell(kr_<)h_\ell^{(1)}(kr_>)Y^*_{\ell m}(\Omega')Y_{\ell m}(\Omega)\,.} \tag{62}$$

It is instructive to take the $k \to 0$ limit of this result. Using eq. (23),

$$j_\ell(kr_<)h_\ell^{(1)}(kr_>) = \frac{-i}{(2\ell+1)k}\,\frac{r_<^\ell}{r_>^{\ell+1}}\big[1+\mathcal{O}(k)\big]\,. \tag{63}$$

Thus, the $k \to 0$ limit of eq. (62) yields,

$$\frac{1}{|\vec{x}-\vec{x}\,'|} = 4\pi\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}\frac{1}{2\ell+1}\,\frac{r_<^\ell}{r_>^{\ell+1}}\;Y^*_{\ell m}(\Omega')Y_{\ell m}(\Omega)\,, \tag{64}$$

in agreement with the result given by eq. (3.70) of Jackson.