1. [Jackson, problem 9.12] An almost spherical surface is defined by

$$R(\theta) = R_0 \left[1 + \beta\, P_2(\cos\theta)\right] \tag{1}$$

has inside of it a uniform volume distribution of charge totaling Q. The small parameter β varies harmonically in time at frequency ω. This corresponds to surface waves on the sphere. Keeping only lowest order terms in β and making the long-wavelength approximation, calculate the nonvanishing multipole moments, the angular distribution of radiation, and the total power radiated.

First, we need to evaluate the charge density $\rho(\vec{\boldsymbol{x}}, t)$. It is a constant ρ_0 for $r \leq R(\theta)$ and zero otherwise. Since the total charge Q is conserved (and hence time independent),

$$Q = \int d^3x\, \rho(\vec{\boldsymbol{x}}, t) = \rho_0 \int r^2\, dr\, d\cos\theta\, d\phi\, \Theta(R(\theta) - r)\,,$$

where the step function is defined as,

$$\Theta(x) = \begin{cases} 1\,, & \text{for } x > 0\,, \\ 0\,, & \text{for } x < 0\,. \end{cases}$$

Thus,

$$Q = 2\pi\rho_0 \int_{-1}^{1} d\cos\theta \int_0^{R(\theta)} r^2\, dr = \frac{2\pi R_0^3 \rho_0}{3} \int_{-1}^{1} d\cos\theta\, \left[1 + \beta\, P_2(\cos\theta)\right]^3\,.$$

Assuming that $|\beta| \ll 1$ and dropping terms of $\mathcal{O}(\beta^2)$, it follows that

$$Q = \frac{2\pi R_0^3 \rho_0}{3} \int_{-1}^{1} d\cos\theta\, \left[1 + 3\beta\, P_2(\cos\theta) + \mathcal{O}(\beta^2)\right] = \frac{4\pi R_0^3 \rho_0}{3} \left[1 + \mathcal{O}(\beta^2)\right]\,,$$

after using the orthogonality relation,

$$\int_{-1}^{1} d\cos\theta\, P_\ell(\cos\theta)\, P_{\ell'}(\cos\theta) = \frac{2}{2\ell + 1}\, \delta_{\ell\ell'}\,.$$

The parameter β varies harmonically with time. Using complex notation,

$$\beta = \beta_0 e^{-i\omega t}$$

Hence, including all terms up to and including $\mathcal{O}(\beta_0)$,

$$\rho(\vec{\boldsymbol{x}}, t) = \frac{3Q}{4\pi R_0^3}\, \Theta(R_0 + R_0\beta_0 P_2(\cos\theta) e^{-i\omega t} - r)\,. \tag{2}$$

Next, we compute the elements of the multipole tensor in the spherical basis,

$$\begin{aligned}
Q_{\ell m}(t) &= \int d^3x\, r^\ell\, Y^*_{\ell m}(\theta,\phi)\,\rho(\vec{\boldsymbol{x}},t) \\
&= \frac{3Q}{4\pi R_0^3}\int d\Omega\, Y^*_{\ell m}(\theta,\phi)\int_0^{R(\theta)} r^{\ell+2}\,dr \\
&= \frac{3QR_0^\ell}{4\pi(\ell+3)}\int d\Omega\, Y^*_{\ell m}(\theta,\phi)\left[1+\beta_0 P_\ell(\cos\theta)e^{-i\omega t}\right]^{\ell+3},
\end{aligned}$$

where $\ell = 1, 2, 3, \ldots$ and $m = -\ell, -\ell+1, \ldots, \ell-1, \ell$. Note that the $\ell = m = 0$ moment does not enter the multipole expansion of the radiation fields. Working to first order in β_0, we can approximate

$$\left[1+\beta_0 P_\ell(\cos\theta)e^{-i\omega t}\right]^{\ell+3} = 1+(\ell+3)\beta_0 P_\ell(\cos\theta)e^{-i\omega t}+\mathcal{O}(\beta_0^2)\,.$$

Writing

$$P_\ell(\cos\theta) = \sqrt{\frac{4\pi}{2\ell+1}}\,Y_{\ell 0}(\theta,\phi)\,,$$

it follows that for $\ell \neq 0$,

$$\begin{aligned}
Q_{\ell m}(t) &= \frac{3QR_0^\ell}{4\pi(\ell+3)}\int d\Omega\, Y^*_{\ell m}(\theta,\phi)\left[1+(\ell+3)\beta_0\, e^{-i\omega t}\sqrt{\frac{4\pi}{5}}Y_{20}(\theta,\phi)\right] \\
&= \frac{3QR_0^\ell}{(\ell+3)\sqrt{4\pi}}\left[\frac{(\ell+3)\beta_0\, e^{-i\omega t}}{\sqrt{5}}\delta_{\ell 2}\,\delta_{m0}\right], \qquad (3)
\end{aligned}$$

after employing the orthogonality relation of the spherical harmonics [cf. eq. (3.55) of Jackson],

$$\int Y^*_{\ell m}(\theta,\phi)\,Y_{\ell' m'}(\theta,\phi)\,d\Omega = \delta_{\ell\ell'}\,\delta_{mm'}\,.$$

Writing $Q_{\ell m}(t) = Q_{\ell m}e^{-i\omega t}$, it follows that

$$Q_{\ell m} = \frac{3Q\beta_0 R_0^2}{\sqrt{20\pi}}\,\delta_{\ell,2}\,\delta_{m,0}\,.$$

That is, the only non-zero electric multipole moment is

$$Q_{20} = \frac{3Q\beta_0 R_0^2}{\sqrt{20\pi}}\,. \qquad (4)$$

An alternative derivation of eq. (4)

Since we are working to first order in β_0, it is convenient to expand the Θ-function that appears in eq. (2) using the fact that $\delta(x) = d\Theta(x)/dx$. Thus, to $\mathcal{O}(\beta_0)$,

$$\rho(\vec{\boldsymbol{x}},t) = \frac{3Q}{4\pi R_0^3}\left[\Theta(R_0-r)+R_0\beta_0 P_2(\cos\theta)\,e^{-i\omega t}\,\delta(R_0-r)\right]. \qquad (5)$$

Using eq. (9.170) of Jackson, we can evaluate the multipole moments for $\ell \neq 0$,

$$
\begin{aligned}
Q_{\ell m}(t) &= \int d^3x\, r^\ell\, Y^*_{\ell m}(\theta,\phi)\, \rho(\vec{\boldsymbol{x}},t) \\
&= \frac{3Q}{4\pi R_0^3} \int d^3x\, r^\ell\, Y^*_{\ell m}(\theta,\phi)\, \left[\Theta(R_0-r) + R_0\beta_0 P_2(\cos\theta)\, e^{-i\omega t}\, \delta(R_0-r)\right] \\
&= \frac{3QR_0}{4\pi R_0^3} \left[\sqrt{\frac{4\pi}{5}}\, R_0^{\ell+3}\beta_0\, e^{-i\omega t}\, \delta_{\ell 2}\, \delta_{m0}\right] , \qquad (6)
\end{aligned}
$$

which reproduces eq. (3).

As for the other possible multipole moments, we first note that there is no magnetization in this problem so that $Q'_{\ell m} = M'_{\ell m} = 0$. [cf. eqs.(9.170) and (9.172) of Jackson]. However, there is a non-zero harmonic current density due to motion of electric charges. The azimuthal symmetry of the problem implies that $\vec{\boldsymbol{J}}(\vec{\boldsymbol{x}},t)$, when written in spherical coordinates, has no $\hat{\boldsymbol{\phi}}$ component and is independent of ϕ.[1] That is,

$$
\vec{\boldsymbol{J}}(\vec{\boldsymbol{x}},t) = \left[J_r(r,\theta)\,\hat{\boldsymbol{n}} + J_\theta(r,\theta)\,\hat{\boldsymbol{\theta}}\right]\, e^{-i\omega t} ,
$$

where $\hat{\boldsymbol{n}} \equiv \vec{\boldsymbol{x}}/r$ is the unit vector in the radial direction. Using eq. (9.172) of Jackson (in SI units), with $\vec{\boldsymbol{J}}(\vec{\boldsymbol{x}},t) = \vec{\boldsymbol{J}}(\vec{\boldsymbol{x}})\, e^{-i\omega t}$,

$$
M_{\ell m} = -\frac{1}{\ell+1} \int d^3x\, r^\ell\, Y^*_{\ell m}(\theta,\phi)\, \vec{\boldsymbol{\nabla}} \boldsymbol{\cdot} (\vec{\boldsymbol{x}} \times \vec{\boldsymbol{J}}(\vec{\boldsymbol{x}})) . \qquad (7)
$$

Using,

$$
\vec{\boldsymbol{x}} \times \vec{\boldsymbol{J}}(\vec{\boldsymbol{x}}) = r\,\hat{\boldsymbol{n}} \times \vec{\boldsymbol{J}}(\vec{\boldsymbol{x}}) = r\, J_\theta(r,\theta)\,\hat{\boldsymbol{\phi}} ,
$$

we conclude that

$$
\vec{\boldsymbol{\nabla}} \boldsymbol{\cdot} (\vec{\boldsymbol{x}} \times \vec{\boldsymbol{J}}(\vec{\boldsymbol{x}})) = \frac{1}{\sin\theta}\,\frac{\partial J_\theta}{d\phi} = 0 .
$$

Hence, it follows from eq. (7) that

$$
M_{\ell m} = 0 .
$$

The angular distribution of the radiated power can be obtained from eqs. (9.151) and (9.169) of Jackson,

$$
\frac{dP}{d\Omega} = \tfrac{1}{2} Z_0 c^2 k^{2\ell+2}\, \frac{\ell+1}{\ell[(2\ell+1)!!]^2}\, |Q_{\ell m}|^2\, |\vec{\boldsymbol{X}}_{\ell m}|^2 , \qquad (8)
$$

where

$$
\vec{\boldsymbol{X}}_{\ell m} = \frac{1}{\sqrt{\ell(\ell+1)}}\, \vec{\boldsymbol{L}}\, Y_{\ell m}(\theta,\phi) ,
$$

is a vector spherical harmonic. Integrating over solid angles is trivial since the $\vec{\boldsymbol{X}}_{\ell m}$ are normalized to unity. Thus,

$$
P = \tfrac{1}{2} Z_0 c^2 k^{2\ell+2}\, \frac{\ell+1}{\ell[(2\ell+1)!!]^2}\, |Q_{\ell m}|^2 . \qquad (9)
$$

[1] An explicit expression for $\vec{\boldsymbol{J}}(\vec{\boldsymbol{x}},t)$ will be given in an added note following this solution.

Inserting the value for Q_{20} obtained in eq. (4) into the above formulae, and noting that

$$|\vec{\boldsymbol{X}}_{20}|^2 = \frac{15}{8\pi}\,\sin^2\theta\cos^2\theta\,,$$

according to Table 9.1 on p. 437 of Jackson, it follows that

$$\frac{dP}{d\Omega} = \frac{3Z_0c^2Q^2\beta_0^2R_0^4k^6}{2000\pi}\,\frac{15}{8\pi}\,\sin^2\theta\cos^2\theta\,,$$

and

$$P = \frac{3Z_0c^2Q^2\beta_0^2R_0^4k^6}{2000\pi}\,.$$

<u>*ADDED NOTE*</u>:

In this added note, we shall obtain an explicit form for $\vec{\boldsymbol{J}}(\vec{\boldsymbol{x}},t)$ which is valid to first order in β. As noted previously, the azimuthal symmetry of the problem implies that $\vec{\boldsymbol{J}}(\vec{\boldsymbol{x}},t)$ has no $\hat{\boldsymbol{\phi}}$ component and is independent of the azimuthal angle ϕ. That is,

$$\vec{\boldsymbol{J}}(\vec{\boldsymbol{x}},t) = \Big[J_r(r,\theta)\hat{\boldsymbol{n}} + J_\theta(r,\theta)\hat{\boldsymbol{\theta}}\Big]\,e^{-i\omega t}\,,$$

where $\hat{\boldsymbol{n}} \equiv \vec{\boldsymbol{x}}/r$ is the unit vector in the radial direction. Using the continuity equation,

$$\vec{\boldsymbol{\nabla}}\cdot\vec{\boldsymbol{J}} + \frac{d\rho}{dt} = 0\,,$$

we can compute $\vec{\boldsymbol{\nabla}}\cdot\vec{\boldsymbol{J}}$ using the result for ρ obtained in eq. (5). Hence,

$$\vec{\boldsymbol{\nabla}}\cdot\vec{\boldsymbol{J}} = -\frac{\partial\rho}{\partial t} = \frac{3i\omega Q\beta_0}{4\pi R_0^2}\,P_2(\cos\theta)\,e^{-i\omega t}\,\delta(R_0-r)\,. \tag{10}$$

In spherical coordinates, we have

$$\vec{\boldsymbol{\nabla}}\cdot\vec{\boldsymbol{J}} = \frac{1}{r^2\sin\theta}\left[\sin\theta\,\frac{\partial}{\partial r}\left(r^2J_r\right) + r\,\frac{\partial}{\partial\theta}\left(\sin\theta\,J_\theta\right) + r\frac{\partial J_\phi}{\partial\phi}\right]\,.$$

Since $\vec{\boldsymbol{J}}$ arises due to charges in motion, it must be proportional to β. Thus, to leading order in β, it must also be true that $\vec{\boldsymbol{J}}$ is proportional to $\Theta(R_0-r)$ since we can drop any β-dependence in the argument of the Θ-function (as the dropped terms will only contribute at higher order in β). Hence, as $J_\phi = 0$, we can write:

$$\vec{\boldsymbol{J}}(\vec{\boldsymbol{x}},t) = \beta_0\,e^{-i\omega t}\,\Theta(R_0-r)\left[J_1\hat{\boldsymbol{n}} + J_2\hat{\boldsymbol{\theta}}\right]\,. \tag{11}$$

Using eq. (11), we compute

$$\begin{aligned}\vec{\boldsymbol{\nabla}}\cdot\vec{\boldsymbol{J}} &= \beta_0\,e^{-i\omega t}\left\{\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\Theta(R_0-r)J_1\right) + \frac{1}{r\sin\theta}\,\Theta(R_0-r)\frac{\partial}{\partial\theta}\left(\sin\theta\,J_2\right)\right\} \\ &= \beta_0\,e^{-i\omega t}\left\{-J_1\,\delta(R_0-r) + \Theta(R_0-r)\left[\frac{1}{r^2}\,\frac{\partial}{\partial r}\left(r^2J_1\right) + \frac{1}{r\sin\theta}\,\frac{\partial}{\partial\theta}\left(\sin\theta\,J_2\right)\right]\right\}\,. \qquad (12)\end{aligned}$$

Comparing this result with eq. (10), we can immediately equate the coefficients of the delta function, which yields

$$J_1 = -\frac{3i\omega Q\beta_0}{4\pi R_0^2}\, P_2(\cos\theta)\,. \tag{13}$$

Inserting this result back into eq. (12), and comparing again with eq. (10), we conclude that the overall coefficient of the step function must vanish. That is,

$$-\frac{3i\omega Q\beta_0}{4\pi R_0^2}\, P_2(\cos\theta)\,\frac{2}{r} + \frac{1}{r\sin\theta}\,\frac{\partial}{\partial\theta}\,(\sin\theta\, J_2) = 0\,.$$

Using $P_2(\cos\theta) = \frac{1}{2}(3\cos^2\theta - 1)$, we obtain

$$-\frac{3i\omega Q\beta_0}{4\pi R_0^2}\,\sin\theta\,\big(3\cos^2\theta - 1\big) + \frac{\partial}{\partial\theta}\,(\sin\theta\, J_2) = 0\,.$$

Hence,

$$J_2\sin\theta = -\frac{3i\omega Q\beta_0}{4\pi R_0^2}\int \big(3\cos^2\theta - 1\big)\, d\cos\theta = \frac{3i\omega Q}{4\pi R_0^2}\,\cos\theta(1-\cos^2\theta)\,,$$

or equivalently,[2]

$$J_2 = \frac{3i\omega Q\beta_0}{8\pi R_0^2}\,\sin 2\theta\,. \tag{14}$$

Inserting eqs. (13) and (14) back into eq. (11) yields the final result, which is valid at first order in β_0,

$$\vec{\boldsymbol{J}}(\vec{\boldsymbol{x}},t) = -\frac{3i\omega Q\beta_0}{8\pi R_0^2}\, e^{-i\omega t}\,\Theta(R_0 - r)\,\Big[(3\cos^2\theta - 1)\hat{\boldsymbol{n}} - \sin 2\theta\,\hat{\boldsymbol{\theta}}\Big]\,.$$

2. [Jackson, problem 9.17] Treat the linear antenna of Jackson, problem 9.16 (on Problem Set 3) by the multipole expansion method.

(a) Calculate the multipole moments (electric dipole, magnetic dipole, and electric quadrupole) exactly and in the long-wavelengths approximation.

As in Jackson, problem 9.16, we shall choose the z-axis to lie along the antenna, and let $z = 0$ correspond to the center of the antenna. Then, $\vec{\boldsymbol{J}}(\vec{\boldsymbol{x}},t) = \vec{\boldsymbol{J}}(\vec{\boldsymbol{x}})\, e^{-i\omega t}$, where

$$\vec{\boldsymbol{J}}(\vec{\boldsymbol{x}},t) = I\sin\left(\frac{2\pi z}{d}\right)\,\delta(x)\,\delta(y)\,\hat{\boldsymbol{z}}\,, \qquad \text{for } |z| \leq \tfrac{1}{2}d\,, \tag{15}$$

where d is the length of the antenna. It is convenient to rewrite this in spherical coordinates. Note that $\hat{\boldsymbol{z}} = \hat{n}$ for $\cos\theta = 1$ (i.e., $\theta = 0$) and $\hat{\boldsymbol{z}} = -\hat{\boldsymbol{n}}$ for $\cos\theta = -1$ (i.e., $\theta = \pi$), where $\hat{\boldsymbol{n}}$

[2] In evaluation of the indefinite integral, the constant of integration must be set to zero, since J_2 must be non-singular at $\theta = 0$ and at $\theta = \pi$.

is a unit vector pointing in the radial direction, and $r \equiv |\vec{\boldsymbol{x}}|$ is the radial coordinate. Hence, we may write[3]

$$\vec{\boldsymbol{J}}(\vec{\boldsymbol{x}}) = \frac{I}{2\pi r^2}\sin\left(\frac{2\pi r}{d}\right)\big[\delta(\cos\theta - 1) + \delta(\cos\theta + 1)\big]\Theta(\tfrac{1}{2}d - r)\hat{\boldsymbol{n}}\,, \tag{16}$$

where I have inserted the Heavyside step function since the current $I(z,t) = 0$ for $|z| > \frac{1}{2}d$. In obtaining eq. (16), I used the fact that

$$\sin\left(\frac{2\pi z}{d}\right) = \sin\left(\frac{2\pi r\varepsilon(z)}{d}\right) = \varepsilon(z)\sin\left(\frac{2\pi r}{d}\right)\,,$$

where the sign function $\varepsilon(z)$ is defined as

$$\varepsilon(z) = \begin{cases} +1\,, & \text{for } z > 0, \\ -1\,, & \text{for } z < 0. \end{cases}$$

Finally, we note that $\hat{\boldsymbol{n}} = \epsilon(z)\hat{\boldsymbol{z}}$ along the z-axis.

We shall make use of eqs. (9.167) and (9.168) of Jackson for the electric and magnetic multipole coefficients. In the absence of magnetization, in MKS units,

$$a_E(\ell, m) = \frac{k^2}{i\sqrt{\ell(\ell+1)}}\int Y^*_{\ell m}(\theta,\phi)\left\{c\rho(\vec{\boldsymbol{x}})\frac{\partial}{\partial r}\big[rj_\ell(kr)\big] + ik\,\vec{\boldsymbol{x}}\boldsymbol{\cdot}\vec{\boldsymbol{J}}(\vec{\boldsymbol{x}})j_\ell(kr)\right\}d^3x\,, \tag{17}$$

$$a_B(\ell, m) = \frac{k^2}{i\sqrt{\ell(\ell+1)}}\int Y^*_{\ell m}(\theta,\phi)\vec{\boldsymbol{\nabla}}\boldsymbol{\cdot}\big(\vec{\boldsymbol{x}}\times\vec{\boldsymbol{J}}(\vec{\boldsymbol{x}})\big)j_\ell(kr)d^3x\,. \tag{18}$$

It is convenient to integrate by parts in evaluating the first term of the integrand in eq. (17). The surface term can be dropped, since the charge density is localized. Since $d^3x = r^2\,dr\,d\Omega$, after integrating by parts, one obtains

$$-\frac{\partial}{\partial r}\big[r^2\rho(\vec{\boldsymbol{x}})\big]\,dr = -\left(\frac{\partial\rho(\vec{\boldsymbol{x}})}{\partial r} + \frac{2}{r}\rho(\vec{\boldsymbol{x}})\right)r^2\,dr\,.$$

It then follows that

$$a_E(\ell, m) = \frac{k^2}{i\sqrt{\ell(\ell+1)}}\int Y^*_{\ell m}(\theta,\phi)j_\ell(kr)\left\{-c\left(2 + r\frac{\partial}{\partial r}\right)\rho(\vec{\boldsymbol{x}}) + ik\,\vec{\boldsymbol{x}}\boldsymbol{\cdot}\vec{\boldsymbol{J}}(\vec{\boldsymbol{x}})\right\}d^3x\,, \tag{19}$$

which is the version obtained in class.[4]

[3]Note that this differs from eq. (9.179) of Jackson by a relative sign. This difference is due to the fact that for the antenna showed in Figure 9.6 of Jackson, we have $I(-z) = I(z)$. In contrast, in this problem, eq. (15) yields $I(-z) = -I(z)$.

[4]One small advantage of using eq. (17) instead of eq. (19) is that no delta functions arise in the computation [cf. eq. (32) below]. By employing eq. (17), Jackson can simply set the limits of the radial integration to $0 \leq r \leq \frac{1}{2}d$, and otherwise ignore the implicit Heavyside step function in his analysis of the linear, centerfed antenna on pp. 445–446.

First consider the computation of $a_B(\ell,m)$. Using the vector identity,

$$\vec{\boldsymbol{\nabla}}\boldsymbol{\cdot}\big(\vec{\boldsymbol{x}}\times\vec{\boldsymbol{J}}\,\big)=\vec{\boldsymbol{J}}\boldsymbol{\cdot}(\vec{\boldsymbol{\nabla}}\times\vec{\boldsymbol{x}})-\vec{\boldsymbol{x}}\boldsymbol{\cdot}(\vec{\boldsymbol{\nabla}}\times\vec{\boldsymbol{J}})=-\vec{\boldsymbol{x}}\boldsymbol{\cdot}(\vec{\boldsymbol{\nabla}}\times\vec{\boldsymbol{J}})\,,$$

after using $\vec{\boldsymbol{\nabla}}\times\vec{\boldsymbol{x}}=0$. However, the current density given in eq. (16) is purely radial, which implies that $\vec{\boldsymbol{\nabla}}\times\vec{\boldsymbol{J}}=0$. Therefore, we conclude that $\vec{\boldsymbol{\nabla}}\boldsymbol{\cdot}\big(\vec{\boldsymbol{x}}\times\vec{\boldsymbol{J}}\big)=0$, which implies that $a_B(\ell,m)=0$. That is, all the magnetic multipole coefficients vanish.

To evaluate the electric multipole coefficients, $a_E(\ell,m)$, we can either use eq. (17) or eq. (19). We shall first employ eq. (17), and then in an addendum we will provide details of the calculation that makes use of eq. (19).

For harmonic sources [cf. eq. (9.15) of Jackson], $\vec{\boldsymbol{\nabla}}\boldsymbol{\cdot}\vec{\boldsymbol{J}}=i\omega\rho$. Using eq. (16), we see that $\vec{\boldsymbol{J}}$ is purely radial, $\vec{\boldsymbol{J}}=J_r\hat{\boldsymbol{n}}$, and

$$\vec{\boldsymbol{\nabla}}\boldsymbol{\cdot}\vec{\boldsymbol{J}}=\frac{1}{r^2}\frac{\partial}{\partial r}\big(r^2J_r\big)=\frac{I}{r^2}\big[\delta(\cos\theta-1)+\delta(\cos\theta+1)\big]$$
$$\times\left\{\frac{1}{d}\cos\left(\frac{2\pi r}{d}\right)\Theta(\tfrac{1}{2}d-r)-\frac{1}{2\pi}\sin\left(\frac{2\pi r}{d}\right)\delta(\tfrac{1}{2}d-r)\right\}.$$

Noting that $\sin(2\pi r/d)\delta(\frac{1}{2}d-r)=\sin\pi\,\delta(\frac{1}{2}d-r)=0$, we can drop the delta function in the previous equation. We conclude that

$$\rho(\vec{\boldsymbol{x}})=\frac{1}{ikc}\vec{\boldsymbol{\nabla}}\boldsymbol{\cdot}\vec{\boldsymbol{J}}(\vec{\boldsymbol{x}})=\frac{I}{ikcr^2d}\cos\left(\frac{2\pi r}{d}\right)\big[\delta(\cos\theta-1)+\delta(\cos\theta+1)\big]\Theta(\tfrac{1}{2}d-r)\,,\tag{20}$$

after making use of $\omega=kc$. We also note that eq. (16) yields

$$\vec{\boldsymbol{x}}\boldsymbol{\cdot}\vec{\boldsymbol{J}}(\vec{\boldsymbol{x}})=\frac{I}{2\pi r}\sin\left(\frac{2\pi r}{d}\right)\big[\delta(\cos\theta-1)+\delta(\cos\theta+1)\big]\Theta(\tfrac{1}{2}d-r)\,.\tag{21}$$

after using $\vec{\boldsymbol{x}}\boldsymbol{\cdot}\hat{\boldsymbol{n}}=r$.

Plugging eqs. (20) and (21) into eq. (17), and evaluating the integral using spherical coordinates, $d^3x=r^2\,dr\,d\Omega$,

$$a_E(\ell,m)=\frac{Ik}{d\sqrt{\ell(\ell+1)}}\int Y^*_{\ell m}(\theta,\phi)\big[\delta(\cos\theta-1)+\delta(\cos\theta+1)\big]d\Omega$$
$$\times\int_0^{d/2}r\,dr\left\{-\cos\left(\frac{2\pi r}{d}\right)\frac{1}{r}\frac{\partial}{\partial r}\big[rj_\ell(kr)\big]+\frac{k^2}{2\pi}\sin\left(\frac{2\pi r}{d}\right)j_\ell(kr)\right\}.\tag{22}$$

We first evaluate the angular integral above. Writing $d\Omega=d\cos\theta\,d\phi$, consider the integral

$$\frac{1}{2\pi}\int Y^*_{\ell m}(\theta,\phi)\big[\delta(\cos\theta-1)+\delta(\cos\theta+1)\big]d\cos\theta\,d\phi\,.$$

Since $Y^*_{\ell m}(\theta,\phi)\propto e^{-im\phi}$, the ϕ integral yields

$$\int_0^{2\pi}e^{-im\phi}d\phi=\delta_{m0}\,.$$

Thus, in light of eq. (3.57) of Jackson and the properties of the Legendre polynomials,

$$\frac{1}{2\pi}\int Y^*_{\ell m}(\theta,\phi)\big[\delta(\cos\theta-1)+\delta(\cos\theta+1)\big]d\cos\theta\, d\phi = \delta_{m0}\big[Y^*_{\ell 0}(0,\phi)+Y^*_{\ell 0,\phi}(\pi)\big]$$

$$= \delta_{m0}\left(\frac{2\ell+1}{4\pi}\right)^{1/2}\big[1+(-1)^\ell\big]. \quad (23)$$

Plugging eq. (23) back into eq. (22),

$$a_E(\ell,m)=\frac{Ik}{d}\sqrt{\frac{\pi(2\ell+1)}{\ell(\ell+1)}}\,\delta_{m0}\big[1+(-1)^\ell\big]\int_0^{d/2}\left\{-\cos\left(\frac{2\pi r}{d}\right)\frac{\partial}{\partial r}\big[rj_\ell(kr)\big]+\frac{k^2rd}{2\pi}\sin\left(\frac{2\pi r}{d}\right)j_\ell(kr)\right\}dr\,.$$

The first term of the integrand above can be rewritten using an integration by parts,

$$\int_0^{d/2}\cos\left(\frac{2\pi r}{d}\right)\frac{\partial}{\partial r}\big[rj_\ell(kr)\big]=\cos\left(\frac{2\pi r}{d}\right)rj_\ell(kr)\Big|_0^{d/2}+\frac{2\pi}{d}\int_0^{d/2}r\sin\left(\frac{2\pi r}{d}\right)j_\ell(kr)dr$$

$$=-\tfrac{1}{2}d\,j_\ell\big(\tfrac{1}{2}kd\big)+\frac{2\pi}{d}\int_0^{d/2}r\sin\left(\frac{2\pi r}{d}\right)j_\ell(kr)dr\,.$$

We then end up with

$$a_E(\ell,m)=\frac{Ik}{2}\sqrt{\frac{2\ell+1}{\pi\ell(\ell+1)}}\,\delta_{m0}\big[1+(-1)^\ell\big]\left\{\pi j_\ell\big(\tfrac{1}{2}kd\big)+\left[k^2-\left(\frac{2\pi}{d}\right)^2\right]\int_0^{d/2}\sin\left(\frac{2\pi r}{d}\right)j_\ell(kr)\,r\,dr\right\}. \quad (24)$$

By assumption, the sinusoidal current makes a full wavelength, which implies that

$$k=\frac{2\pi}{d}\,. \quad (25)$$

Hence, after setting $kd=2\pi$ in eq. (24), we arrive at the final result,

$$a_E(\ell,m)=\frac{Ik}{2}\sqrt{\frac{\pi(2\ell+1)}{\ell(\ell+1)}}\delta_{m0}\big[1+(-1)^\ell\big]j_\ell(\pi)\,. \quad (26)$$

We now consider the long wavelength approximation, $kd\ll 1$. We will do the computation in two ways. First we will start with eq. (24) and use the small argument approximation for the spherical Bessel function,

$$j_\ell(kr)\simeq\frac{(kr)^\ell}{(2\ell+1)!!}\,.$$

Changing variables to $x\equiv 2r/d$,

$$a_E(\ell,m)\simeq\frac{Ik}{2(2\ell+1)!!}\sqrt{\frac{\pi(2\ell+1)}{\ell(\ell+1)}}\delta_{m0}\big[1+(-1)^\ell\big]\left(\frac{kd}{2}\right)^\ell\left\{1-\pi\int_0^1 x^{\ell+1}\sin(\pi x)dx\right\},$$

after dropping the $\mathcal{O}(k^2)$ term in the factor that multiplies the integral in eq. (24). Integrating by parts yields

$$\pi \int_0^1 x^{\ell+1} \sin(\pi x) dx = 1 + (\ell+1) \int_0^1 x^\ell \cos(\pi x) dx \,.$$

Hence, we obtain a slightly simpler result,

$$a_E(\ell,m) \simeq -\frac{Ik}{2(2\ell+1)!!}\sqrt{\frac{\pi(\ell+1)(2\ell+1)}{\ell}}\delta_{m0}\big[1+(-1)^\ell\big]\left(\frac{kd}{2}\right)^\ell \int_0^1 x^\ell \cos(\pi x) dx \,. \tag{27}$$

As a check of eq. (27), we can perform thee computation using eqs. (9.169)–(9.170) of Jackson (after setting the magnetization to zero),

$$a_E(\ell,m) \simeq \frac{ck^{\ell+2}}{i(2\ell+1)!!}\left(\frac{\ell+1}{\ell}\right)^{1/2} Q_{\ell m} \,, \tag{28}$$

where

$$Q_{\ell m} = \int r^\ell Y^*_{\ell m}(\theta,\phi)\rho(\vec{\boldsymbol{x}})\, d^3x \,. \tag{29}$$

Inserting eq. (20) into eq. (29), and making use of eq. (23),

$$Q_{\ell m} = \frac{I\sqrt{\pi(2\ell+1)}}{ikcd}\delta_{m0}\big[1+(-1)^\ell\big]\int_0^{d/2} r^\ell \cos\left(\frac{2\pi r}{d}\right) dr \,,$$

after using $\omega = kc$. Changing variables to $x = 2r/d$,

$$Q_{\ell m} = \frac{I\sqrt{\pi(2\ell+1)}}{2ikc}\delta_{m0}\left(\frac{d}{2}\right)^\ell \big[1+(-1)^\ell\big]\int_0^1 x^\ell \cos(\pi x) dx \,.$$

Plugging this result into eq. (28) yields

$$a_E(\ell,m) \simeq -\frac{Ik}{2(2\ell+1)!!}\sqrt{\frac{\pi(\ell+1)(2\ell+1)}{\ell}}\delta_{m0}\big[1+(-1)^\ell\big]\left(\frac{kd}{2}\right)^\ell \int_0^1 x^\ell \cos(\pi x) dx \,,$$

in agreement with eq. (27)

We now evaluate these results explicitly for the electric dipole ($\ell = 1$) and the electric quadrupole ($\ell = 2$). Due to the factor of $1 + (-1)^\ell$, we immediately see that only even ℓ multipoles survive. Hence, the electric dipole coefficient vanishes. Thus, we henceforth focus on the electric quadrupole coefficient. First, we use the exact result given in eq. (26). Using

$$j_2(x) = \left(\frac{3}{x^3} - \frac{1}{x}\right)\sin x - \frac{3}{x^2}\cos x \,,$$

it follows that $j_2(\pi) = 3/\pi^2$. Hence, for $kd = 2\pi$ and $\ell = 2$, eq. (26) yields

$$a_E(2,0) = Ik\sqrt{\frac{15}{2\pi^3}} \,. \tag{30}$$

Let us compare this result with eq. (27), which was obtained in the long wavelength approximation.

$$a_E(2,0) \simeq -Ik\sqrt{\frac{\pi}{30}}\left(\frac{kd}{2}\right)^2 \int_0^1 x^2\cos(\pi x)dx\,.$$

Performing the integral,

$$\int_0^1 x^2\cos(\pi x)dx = \frac{1}{\pi^3}\big[2\pi x\cos(\pi x) + (\pi^2x^2-2)\sin(\pi x)\big]\bigg|_0^1 = -\frac{2}{\pi^2}\,,$$

we end up with

$$a_E(2,0) \simeq Ik\sqrt{\frac{2}{15\pi^3}}\left(\frac{kd}{2}\right)^2\,.$$

This result should only be valid for $kd \ll 1$. Nevertheless, to compare with eq. (30), we bravely put $kd = 2\pi$ to obtain

$$a_E(2,0) \simeq Ik\sqrt{\frac{2\pi}{15}}\,, \tag{31}$$

which is larger than the exact result given in eq. (30) by a factor of $2\pi^2/15 \simeq 1.316$. Not too bad!

<u>*ADDENDUM*</u>:

As promised, we exhibit the necessary calculations to obtain $a_E(\ell,m)$ starting from eq. (19). In this method, one needs to keep track of the Heavyside step function, since it will generate a delta function when computing $\partial\rho/\partial r$ that cannot be ignored, as noted in footnote 4.

In this method, we use eq. (20) to compute

$$-\left(2+r\frac{\partial}{\partial r}\right)\rho(\vec{\boldsymbol{x}}) = \big[\delta(\cos\theta-1)+\delta(\cos\theta+1)\big]\frac{I}{ickrd}\left\{\frac{2\pi}{d}\sin\left(\frac{2\pi r}{d}\right)\Theta(\tfrac{1}{2}d-r)+\cos\left(\frac{2\pi r}{d}\right)\delta(r-\tfrac{1}{2}d)\right\}.$$

The delta function piece can be simplified by using $\cos(2\pi r/d)\delta(r-\frac{1}{2}d) = \cos\pi\,\delta(r-\frac{1}{2}d) = -\delta(r-\frac{1}{2}d)$. Hence,

$$-\left(2+r\frac{\partial}{\partial r}\right)\rho(\vec{\boldsymbol{x}}) = \big[\delta(\cos\theta-1)+\delta(\cos\theta+1)\big]\frac{I}{ickrd}\left\{\frac{2\pi}{d}\sin\left(\frac{2\pi r}{d}\right)\Theta(\tfrac{1}{2}d-r)-\delta(r-\tfrac{1}{2}d)\right\}. \tag{32}$$

Using eq. (21), we end up with

$$\begin{aligned}-c\left(2+r\frac{\partial}{\partial r}\right)\rho + ik\vec{\boldsymbol{x}}\boldsymbol{\cdot}\vec{\boldsymbol{J}} = \frac{2\pi I}{ikrd^2}&\left\{\left[1-\left(\frac{kd}{2\pi}\right)^2\right]\sin\left(\frac{2\pi r}{d}\right)\Theta(\tfrac{1}{2}d-r) - \frac{d}{2\pi}\delta(r-\tfrac{1}{2}d)\right\}\\ &\times\big[\delta(\cos\theta-1)+\delta(\cos\theta+1)\big]\,.\end{aligned} \tag{33}$$

We now insert eq. (33) into eq. (19). Using eq. (23), and performing some algebraic simplifications, it follows that

$$a_E(\ell,m) = \frac{Ik}{2}\sqrt{\frac{2\ell+1}{\pi\ell(\ell+1)}}\delta_{m0}\big[1+(-1)^\ell\big]\left\{\pi j_\ell(\tfrac{1}{2}kd)+\left[k^2-\left(\frac{2\pi}{d}\right)^2\right]\int_0^{d/2}\sin\left(\frac{2\pi r}{d}\right)j_\ell(kr)\,r\,dr\right\},$$

which reproduces eq. (24).

(b) Compare the shape of the angular distribution of the radiated power for the lowest nonvanishing multipole with the exact distribution obtained in Jackson, problem 9.16 (on Problem Set 3)

Using eq. (9.151) of Jackson, the angular distribution of power for a pure electric multipole of order (ℓ, m) is given by,

$$\frac{dP(\ell,m)}{d\Omega} = \frac{Z_0}{2k^2}|a_E(\ell,m)|^2|\vec{\boldsymbol{X}}_{\ell m}|^2\,.$$

We apply this result to the exact form of the pure electric multipole of order $(\ell, m) = (2, 0)$ obtained in eq. (30), which we rewrite again here,

$$a_E(2,0) = Ik\sqrt{\frac{15}{2\pi^3}}\,.$$

Using Table 9.1 on p. 437 of Jackson,

$$|\vec{\boldsymbol{X}}_{\ell m}|^2 = \frac{15}{8\pi}\sin^2\theta\cos^2\theta\,,$$

Hence,

$$\frac{dP(2,0)}{d\Omega} = \frac{225Z_0I^2}{32\pi^4}\sin^2\theta\cos^2\theta\,. \tag{34}$$

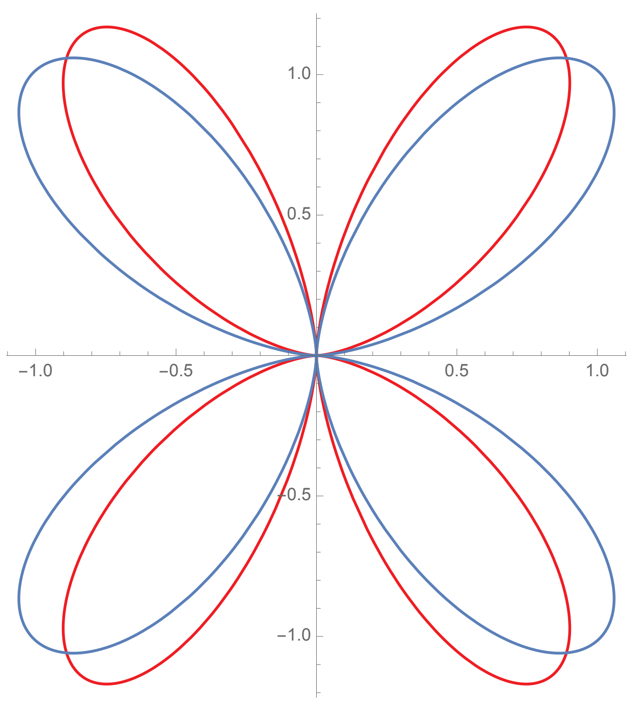

Figure 1: A polar plot of the antenna pattern of a thin linear antenna with a sinusoidal current that makes a full wavelength of oscillation. Normalization has been chosen such that $Z_0I^2 = 8\pi^2$. The angular distribution of the radiated power, shown in red, is given by eq. (35). This is compared with the corresponding angular distribution of the electric quadrupole component, shown in blue, which is given by eq. (34). This plot was created with Mathematica 11 software.

This should be compared with the exact result,

$$\frac{dP}{d\Omega} = \frac{Z_0 I^2}{8\pi^2}\left[\frac{\sin(\pi\cos\theta)}{\sin\theta}\right]^2 . \tag{35}$$

obtained in Jackson, problem 9.16.

(c) Determine the total power radiated for the lowest multipole and the corresponding radiation resistance using both multipole moments from part (a). Compare with part (b) of Jackson, problem 9.16. Is there a paradox here?

The total power radiated by a pure electric multipole of order (ℓ, m) is given by eq. (9.154) of Jackson,

$$P(\ell, m) = \frac{Z_0}{2k^2}|a(\ell, m)|^2 .$$

In part (b) we obtained two expressions for $a_E(2, 0)$. The first expression was exact for $kd = 2\pi$ [cf. eq. (30)],

$$a_E(2, 0) = Ik\sqrt{\frac{15}{2\pi^3}} . \tag{36}$$

The second was computed in the long-wavelength limit, but with $kd = 2\pi$ [cf. eq. (31)],

$$a_E(2, 0) \simeq Ik\sqrt{\frac{2\pi}{15}} . \tag{37}$$

If we use the exact electric quadrupole result [eq. (36)], then we obtain

$$P(2, 0) = \frac{15 Z_0 I^2}{4\pi^3} .$$

The corresponding radiative resistance (in ohms) is equal to the coefficient of $\frac{1}{2}I^2$ [cf. the text below eq. (9.29) of Jackson]. Thus, using $Z_0 = 376.7$ ohms [given below eq. (7.11)′ of Jackson],

$$R_{\rm rad} = \frac{15 Z_0}{2\pi^3} = 91.1 \text{ ohms} , \tag{38}$$

which is remarkably close to the exact result,

$$R_{\rm rad} = (3.114)\,\frac{Z_0}{4\pi} = 93.3 \text{ ohms} \qquad [\text{exact result}] , \tag{39}$$

obtained in part (b) of Jackson, problem 9.16. In contrast, had we used eq. (37), we would have obtained $R_{\rm rad} = 2\pi Z_0/15 = 157.8$ ohms, which is a terrible approximation, as one might have expected.

There is no paradox here. The discussion in Jackson on pp. 446–448 makes clear that keeping the lowest nonvanishing multipole but computing it exactly (i.e., without assuming that $kd \ll 1$) yields an accurate result to the exact antenna problem even for values of kd as large as 2π. Presumably, if one computes the next non-trivial multipole (in this problem, that wold be $\ell = 4$) its numerical contribution, the result would be a rather small correction to the power even when $kd = 2\pi$.

Perhaps the paradox that Jackson is alluding to is based on the expectation that,

$$P(2,0) < P_{\rm exact}\,,$$

since according to eq. (9.155) of Jackson, the total power is equal to an incoherent sum of contributions from all the multipoles. Indeed in our computations above, we did confirm that $P(2,0) < P_{\rm exact}$, or equivalently the radiation resistance of the electric quadrupole contribution given in eq. (38) is less than the exact result obtained in eq. (39). In contrast, the opposite (incorrect) conclusion would have been drawn had we used the expression for $P(2,0)$ based on setting $kd = 2\pi$ in the long wavelength limit [e.g., eq. (37)]. Of course, this latter result is an artifact of a poor approximation.

3. [Jackson, problem 12.3] A particle with mass m and charge e moves in a uniform, static, electric field $\vec{\boldsymbol{E}}_0$.

(a) Solve for the velocity and position of the particle as explicit functions of time, assuming that the initial velocity $\vec{\boldsymbol{v}}_0$ was perpendicular to the electric field.

Using eqs. (12.1) and (12.2) of Jackson and setting $\vec{\boldsymbol{B}} = 0$, we have:

$$\frac{d\vec{\boldsymbol{p}}}{dt} = e\vec{\boldsymbol{E}}\,, \qquad\qquad \frac{dW}{dt} = e\vec{\boldsymbol{v}}\boldsymbol{\cdot}\vec{\boldsymbol{E}}\,,$$

where W is the total mechanical energy (usually called E, but we have renamed this W in order to better distinguish it from the electric field) and $\vec{\boldsymbol{v}}$ is the particle velocity (which is denoted as $\vec{\boldsymbol{u}}$ by Jackson).

Clearly, the motion takes place in a plane containing the $\vec{\boldsymbol{E}}$-field. Without loss of generality, we assume that

$$\vec{\boldsymbol{E}} = E\hat{\boldsymbol{x}}\,,$$

and assume that the motion takes place in the x–y plane. By assumption, $\vec{\boldsymbol{v}}\boldsymbol{\cdot}\vec{\boldsymbol{E}} = 0$ at $t = 0$, in which case $p_x = 0$ at $t = 0$. Solving the equations,

$$\frac{dp_x}{dt} = eE\,, \qquad\qquad \frac{dp_y}{dt} = 0\,, \tag{40}$$

in follows that

$$p_x = eEt\,, \qquad\qquad p_y = p_0\,,$$

where p_0 is a constant.

Using $\vec{\boldsymbol{p}} = \gamma m\vec{\boldsymbol{v}}$ and $W = \gamma mc^2$, it follows that[5]

$$\vec{\boldsymbol{v}} = \frac{c^2\vec{\boldsymbol{p}}}{W} = \frac{c^2\vec{\boldsymbol{p}}}{\sqrt{|\vec{\boldsymbol{p}}|^2c^2 + m^2c^4}}\,. \tag{41}$$

[5]Normally, we write the relativistic energy is given by $E = \gamma mc^2$. However, to avoid confusion with the electric field, I have denoted the relativistic energy by W.

Hence,

$$v_x = \frac{c^2 eEt}{\sqrt{(p_0^2 + e^2E^2t^2)c^2 + m^2c^4}}\,, \qquad v_y = \frac{c^2 p_0}{\sqrt{(p_0^2 + e^2E^2t^2)c^2 + m^2c^4}}\,. \tag{42}$$

Since $\vec{\boldsymbol{v}} = d\vec{\boldsymbol{x}}/dt$, it follows that

$$x = c^2 eE \int \frac{tdt}{\sqrt{W_0^2 + (ceEt)^2}}\,, \qquad y = c^2 p_0 \int \frac{dt}{\sqrt{W_0^2 + (ceEt)^2}}\,, \tag{43}$$

where $W_0^2 = p_0^2c^2 + m^2c^4$.

We shall define the origin of the coordinate system to coincide with $t = 0$. Then computing the integrals in eq. (43) yields

$$x(t) = \frac{1}{eE}\left[\sqrt{W_0^2 + (ceEt)^2} - W_0\right]\,, \qquad y(t) = \frac{p_0 c}{eE}\,\sinh^{-1}\left(\frac{ceEt}{W_0}\right)\,. \tag{44}$$

<u>*REMARKS*</u>:

There is some temptation to first derive a differential equation for $\vec{\boldsymbol{v}}$ before attempting a solution. For example, starting from $\vec{\boldsymbol{v}} = c^2\vec{\boldsymbol{p}}/W$ [cf. eq. (41)], it follows that

$$\frac{d\vec{\boldsymbol{v}}}{dt} = \frac{c^2}{W}\frac{d\vec{\boldsymbol{p}}}{dt} - \frac{c^2\vec{\boldsymbol{p}}}{W^2}\frac{dW}{dt} = \frac{ec^2}{W}\vec{\boldsymbol{E}} - \frac{ec^2\vec{\boldsymbol{p}}}{W^2}\vec{\boldsymbol{v}}\boldsymbol{\cdot}\vec{\boldsymbol{E}}\,. \tag{45}$$

Using $\vec{\boldsymbol{p}} = \gamma m\vec{\boldsymbol{v}}$ and $E = \gamma mc^2$, we obtain

$$\frac{d\vec{\boldsymbol{v}}}{dt} = \frac{e}{\gamma m}\left[\vec{\boldsymbol{E}} - \frac{\vec{\boldsymbol{v}}}{c}\left(\frac{\vec{\boldsymbol{v}}}{c}\boldsymbol{\cdot}\vec{\boldsymbol{E}}\right)\right]\,. \tag{46}$$

In terms of the x and y components of the velocity, eq. (46) is equivalent to:

$$\frac{dv_x}{dt} = \frac{eE}{\gamma m}\left(1 - \frac{v_x^2}{c^2}\right)\,, \tag{47}$$

$$\frac{dv_y}{dt} = -\frac{eEv_xv_y}{\gamma mc^2}\,, \tag{48}$$

where

$$\gamma \equiv \left(1 - \frac{v_x^2 + v_y^2}{c^2}\right)^{-1/2}\,, \tag{49}$$

subject to the boundary condition $v_x(t = 0) = 0$ and $v_y(t = 0) \equiv v_0$.

If we were tasked to solve eqs. (47) and (48), it might not be obvious how to proceed. However, in light of the solution to eq. (40), the method is clear. Namely, we can multiply eqs. (47) and (48) by γ and use

$$\gamma\frac{d\vec{\boldsymbol{v}}}{dt} = \frac{d}{dt}(\gamma\vec{\boldsymbol{v}}) - \vec{\boldsymbol{v}}\,\frac{d\gamma}{dt}\,. \tag{50}$$

Next we would make use of

$$\frac{d\gamma}{dt} = \frac{d}{dt}\left(1 - \frac{\vec{\boldsymbol{v}}\cdot\vec{\boldsymbol{v}}}{c^2}\right)^{-1/2} = \frac{\gamma^3}{c^2}\vec{\boldsymbol{v}}\cdot\frac{d\vec{\boldsymbol{v}}}{dt} = \frac{e}{mc^2}\vec{\boldsymbol{v}}\cdot\vec{\boldsymbol{E}} = \frac{eEv_x}{mc^2}\,, \tag{51}$$

after employing eq. (46) in the penultimate step above. Thus, eqs. (47) and (48) yield

$$\frac{d}{dt}(\gamma v_x) = v_x\frac{d\gamma}{dt} + \frac{eE}{m}\left(1 - \frac{v_x^2}{c^2}\right) = \frac{eE}{m}\,, \tag{52}$$

$$\frac{d}{dt}(\gamma v_y) = v_y\frac{d\gamma}{dt} - \frac{eEv_xv_y}{mc^2} = 0\,, \tag{53}$$

subject to the boundary condition $v_x(t=0) = 0$ and $v_y(t=0) \equiv v_0$. Of course, we have simply reproduced eq. (40). For completeness, we can now trivially solve eqs. (52) and (53):

$$\gamma v_x = \frac{eEt}{m}\,, \qquad\qquad \gamma v_y = \gamma_0 v_0 = \frac{p_0}{m}\,, \tag{54}$$

where $\gamma_0 \equiv (1 - v_0^2/c^2)^{-1/2}$ and we have taken the constant of integration to be $\gamma_0 v_0 \equiv p_0/m$, which defines p_0. Squaring these two equations and adding, we obtain

$$\gamma^2 v^2 = \frac{p_0^2 + e^2E^2t^2}{m^2}\,, \tag{55}$$

where $v^2 = v_x^2 + v_y^2$. Inserting $\gamma^2 v^2 = c^2(\gamma^2 - 1)$ above, it follows that

$$\gamma = \frac{\sqrt{m^2c^2 + p_0^2 + e^2E^2t^2}}{mc}\,. \tag{56}$$

Plugging eq. (56) into eq. (54), we recover the expressions for v_x and v_y previously obtained in eq. (42).

(b) Eliminate the time to obtain the trajectory of the particle in space. Discuss the shape of the path for short and long times (define "short" and "long" times).

We can eliminate t from eq. (44),

$$t = \frac{W_0}{ceE}\sinh\left(\frac{eEy}{p_0c}\right)\,.$$

Inserting this into the equation for $x(t)$ and using the identity $\cosh^2 z - \sinh^2 z = 1$, it follows that

$$x = \frac{W_0}{eE}\left[\cosh\left(\frac{eEy}{p_0c}\right) - 1\right]\,,$$

which is the equation for a catenary curve.

To describe the shape of the path for short and long times, we note that $W_0/(ceE)$ has units of time. This we can define short and long times relative to this quantity. For $t \ll W_0/(ceE)$, we have

$$\sqrt{W_0^2 + (ceEt)^2} \simeq W_0 + \frac{(ceEt)^2}{2W_0}\,, \qquad\qquad \sinh^{-1}\left(\frac{ceEt}{W_0}\right) \simeq \frac{ceEt}{W_0}\,.$$

Hence the approximate form of eq. (44) is

$$x(t) \simeq \frac{c^2 eEt^2}{2W_0}\,, \qquad\qquad y(t) \simeq \frac{p_0 c^2 t}{W_0}\,.$$

Solving for t and inserting the result back into the above equations yields

$$x \simeq \frac{eEW_0 y^2}{2p_0^2 c^2}\,.$$

Since $v_0 = c^2 p_0/W_0$, we can eliminate W_0 from the above expression to obtain,

$$x \simeq \frac{eEy^2}{2p_0 v_0}\,. \tag{57}$$

That is, as short times, the motion is parabolic.[6]

For $t \gg W_0/(ceE)$, eq. (44) yields:

$$x(t) \simeq ct\,, \qquad\qquad y(t) \simeq \frac{p_0 c}{eE}\ln\left(\frac{2ceEt}{W_0}\right)\,.$$

In the latter case, we used:

$$\sinh^{-1} z = \ln\left(z + \sqrt{z^2+1}\right) \simeq \ln 2z\,, \quad \text{for } z \gg 1\,.$$

Hence, to a good approximation,

$$y \simeq \frac{p_0 c}{eE}\ln\left(\frac{2eEx}{W_0}\right)\,,$$

or equivalently,

$$x \simeq \frac{W_0}{2eE}\exp\left(\frac{eEy}{p_0 c}\right)\,.$$

That is, at long times the motion is exponential.

4. [Jackson, problem 12.9] The magnetic field of the earth can be represented approximately by a magnetic dipole of magnetic moment $M = 8.1 \times 10^{25}$ gauss-cm^3. Consider the motion of energetic electrons in the neighborhood of the earth under the action of this dipole field (Van Allen electron belts). [Note that $\vec{\boldsymbol{M}}$ points south.]

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[6] The result of eq. (57) also coincides with the non-relativistic limit (in which case $p_0 = mv_0$). To verify this assertion, we can perform a formal expansion in powers of $1/c$. In this limit, $W_0 \simeq mc^2$ and

$$t \ll \frac{W_0}{ceE} \simeq \frac{mc}{eE}\,,$$

which is always true in the limit of $c \to \infty$ (which is equivalent to taking the non-relativistic limit).

(a) Show that the equation for a line of magnetic force is $r = r_0 \sin^2\theta$, where θ is the usual polar angle (colatitude) measured from the axis of the dipole, and find an expression for the magnitude of $\vec{\boldsymbol{B}}$ along any line of force as a function of θ.

Let the z-axis point from the origin in the direction of the north pole. Then, the magnetic dipole moment (which points south) is given by $\vec{\boldsymbol{M}} = -M\hat{\boldsymbol{z}}$, where $M \equiv |\vec{\boldsymbol{M}}|$. The vector potential is given in gaussian units by:

$$\vec{\boldsymbol{A}}(\vec{\boldsymbol{x}}) = \frac{\vec{\boldsymbol{M}} \times \vec{\boldsymbol{x}}}{|\vec{\boldsymbol{x}}|^3} = \frac{1}{r^3}\det\begin{pmatrix} \hat{\boldsymbol{x}} & \hat{\boldsymbol{y}} & \hat{\boldsymbol{z}} \\ 0 & 0 & -M \\ r\sin\theta\cos\phi & r\sin\theta\sin\phi & r\cos\theta \end{pmatrix}$$

$$= \frac{M\sin\theta}{r^2}\left(\hat{\boldsymbol{x}}\,\sin\phi - \hat{\boldsymbol{y}}\,\cos\phi\right) = -\frac{M\sin\theta}{r^2}\,\hat{\boldsymbol{\phi}}\,,$$

where $r \equiv |\vec{\boldsymbol{x}}|$. Then,

$$\vec{\boldsymbol{B}} = \vec{\boldsymbol{\nabla}} \times \vec{\boldsymbol{A}} = \frac{1}{r^2\sin\theta}\det\begin{pmatrix} \hat{\boldsymbol{r}} & r\hat{\boldsymbol{\theta}} & r\sin\theta\hat{\boldsymbol{\phi}} \\ \dfrac{\partial}{\partial r} & \dfrac{\partial}{\partial\theta} & \dfrac{\partial}{\partial\phi} \\ 0 & 0 & r\sin\theta\,A_\phi \end{pmatrix}\,,$$

where $A_\phi = -M\sin\theta/r^2$. Evaluating the above determinant yields:

$$\vec{\boldsymbol{B}} = -\frac{2M}{r^3}\,\cos\theta\,\hat{\boldsymbol{r}} - \frac{M\sin\theta}{r^3}\,\hat{\boldsymbol{\theta}}\,. \tag{58}$$

Given the magnetic field at every point in space, $\vec{\boldsymbol{B}}(\vec{\boldsymbol{x}})$, one can consider a related vector field, $\vec{\boldsymbol{F}}(\vec{\boldsymbol{x}}) = q_m\vec{\boldsymbol{B}}(\vec{\boldsymbol{x}})$, which gives the force on a magnetic test charge q_m due to the magnetic field at the point $\vec{\boldsymbol{x}}$.[7] If we choose our test charge to have $q_m = 1$, then there is no distinction between the "lines of magnetic force" and the "magnetic field lines." We choose to follow this convention in what follows.

The lines of force follow a curve $\vec{\boldsymbol{x}}(s)$, where the arclength s parameterizes the location along the curve. By definition $\vec{\boldsymbol{B}}(\vec{\boldsymbol{x}})$ is tangent to the lines of force. That is,

$$\frac{d\vec{\boldsymbol{x}}}{ds} = \frac{\vec{\boldsymbol{B}}\bigl(\vec{\boldsymbol{x}}(s)\bigr)}{B}\,, \tag{59}$$

where $B \equiv |\vec{\boldsymbol{B}}|$. To understand the normalization on the right hand side above, we note that eq. (59) is equivalent to the three equations,

$$\frac{dx}{ds} = \frac{B_x}{B}\,, \qquad \frac{dy}{ds} = \frac{B_y}{B}\,, \qquad \frac{dz}{ds} = \frac{B_z}{B}\,.$$

[7]Of course, magnetic charges do not exist in classical electromagnetism. But the concept of "lines of force" was developed before this fact was understood. In the case of the electric field, we do have $\vec{\boldsymbol{F}} = q\vec{\boldsymbol{E}}$, so the terminology "lines of force" makes sense. In the case of magnetic fields, it would be better to refer to the lines of force as the magnetic field lines. Nevertheless, following Jackson, we retain the old terminology in this problem.

Squaring each equation and summing the three resulting equations yields

$$(ds)^2 = (dx)^2 + (dy)^2 + (dz)^2 \,,$$

which is the well-known formula for the differential arclength.

It is convenient to work in spherical coordinates. Consider an infinitesimal displacement $d\vec{\boldsymbol{x}}$, where

$$\vec{\boldsymbol{x}} = r\sin\theta\cos\phi\,\hat{\boldsymbol{x}} + r\sin\theta\sin\phi\,\hat{\boldsymbol{y}} + r\cos\theta\,\hat{\boldsymbol{z}} \,.$$

By the chain rule,

$$\begin{aligned}
d\vec{\boldsymbol{x}} &= \frac{\partial\vec{\boldsymbol{x}}}{dr}\,dr + \frac{\partial\vec{\boldsymbol{x}}}{d\theta}\,d\theta + \frac{\partial\vec{\boldsymbol{x}}}{d\phi}\,d\phi \\
&= (\cos\phi\sin\theta\,\hat{\boldsymbol{x}} + \sin\theta\sin\phi\,\hat{\boldsymbol{y}} + \cos\theta\,\hat{\boldsymbol{z}})\,dr + r\,(\cos\theta\cos\phi\,\hat{\boldsymbol{x}} + \cos\theta\sin\phi\,\hat{\boldsymbol{y}} - \sin\theta\,\hat{\boldsymbol{z}})\,d\theta \\
&\qquad\qquad + r\,(-\sin\theta\sin\phi\,\hat{\boldsymbol{x}} + \sin\theta\cos\phi\,\hat{\boldsymbol{y}})\,d\phi \\
&= \hat{\boldsymbol{r}}\,dr + \hat{\boldsymbol{\theta}}\,r\,d\theta + \hat{\boldsymbol{\phi}}\,r\sin\theta\,d\phi \,.
\end{aligned} \tag{60}$$

The tangent to the curve $\vec{\boldsymbol{x}}(s)$ then takes the form

$$\frac{d\vec{\boldsymbol{x}}}{ds} = \hat{\boldsymbol{r}}\,\frac{dr}{ds} + \hat{\boldsymbol{\theta}}\,r\,\frac{d\theta}{ds} + \hat{\boldsymbol{\phi}}\,r\sin\theta\,\frac{d\phi}{ds} \,. \tag{61}$$

Using eq. (58), it follows that the line of magnetic force is determined by the equation,

$$\frac{d\vec{\boldsymbol{x}}}{ds} = \frac{\vec{\boldsymbol{B}}\,(\vec{\boldsymbol{x}}(s))}{B} = -\frac{2M}{Br^3}\,\cos\theta\,\hat{\boldsymbol{r}} - \frac{M\sin\theta}{Br^3}\,\hat{\boldsymbol{\theta}} \,, \tag{62}$$

where r, θ and ϕ are functions of s. Equating eqs. (61) and (62) yields three differential equations,

$$\frac{dr}{ds} = -\frac{2M\cos\theta}{Br^3} \,, \qquad\qquad r\,\frac{d\theta}{ds} = -\frac{M\sin\theta}{Br^3} \,, \qquad\qquad \frac{d\phi}{ds} = 0 \,. \tag{63}$$

Dividing the first two equations above yields,

$$\frac{dr}{d\theta} = \frac{2r\cos\theta}{\sin\theta} \,,$$

which is easily integrated,

$$\int\frac{dr}{r} = 2\int\frac{\cos\theta}{\sin\theta}\,d\theta \,,$$

Evaluating the integrals and imposing the condition $r = r_0$ at $\theta = \frac{1}{2}\theta$, we obtain

$$\ln\left(\frac{r}{r_0}\right) = 2\ln\sin\theta \,,$$

or equivalently

$$r = r_0\sin^2\theta \,, \tag{64}$$

which we identify as the equation for the line of magnetic force. Note that the third equation in eq. (63) implies that ϕ is a constant along the line of magnetic force.

Finally, we evaluate the magnitude of $\vec{\boldsymbol{B}}$ along the line of force. Since

$$B \equiv |\vec{\boldsymbol{B}}| = \sqrt{B_r^2 + B_\theta^2 + B_\phi^2} = \frac{M}{r^3}\sqrt{4\cos^2\theta + \sin^2\theta}\,,$$

We simply plug in eq. (64) to obtain B as a function of θ along the line of magnetic force,

$$B(\theta) = \frac{M}{r_0^3}\,\frac{\sqrt{1+3\cos^2\theta}}{\sin^6\theta}\,, \tag{65}$$

after using $\sin^2\theta = 1 - \cos^2\theta$ in the numerator above.

(b) A positively charged particle circles around a line of force in the equatorial plane with a gyration radius a and a mean radius R (where $a \ll R$). Show that the particle's azimuthal position (east longitude) changes approximately linearly in time according to:

$$\phi(t) = \phi_0 - \frac{3}{2}\left(\frac{a}{R}\right)^2 \omega_B(t - t_0)\,.$$

where ω_B is the frequency of gyration at radius R.

Assuming that $a \ll R$, we can use eq. (12.55) of Jackson to obtain an approximate formula for the gradient drift velocity,

$$\frac{\vec{\boldsymbol{v}}_G}{\omega_B a} = \frac{a}{2B^2}\left(\vec{\boldsymbol{B}} \times \vec{\boldsymbol{\nabla}}_\perp B\right)\,, \tag{66}$$

where a is the gyration radius and $\vec{\boldsymbol{B}}$ is the field at the equator ($\theta = \frac{1}{2}\pi$). Using eq. (58), this means that

$$\vec{\boldsymbol{B}} = -\hat{\boldsymbol{\theta}}\,\frac{M}{r^3}\bigg|_{r=R} = -\frac{M}{R^3}\,\hat{\boldsymbol{\theta}}\,, \qquad\qquad \vec{\boldsymbol{\nabla}}_\perp B = \hat{\boldsymbol{r}}\,\frac{\partial B}{\partial r}\bigg|_{r=R} = -\frac{3M}{R^4}\,\hat{\boldsymbol{r}}\,, \tag{67}$$

where R is the mean radius. In computing $\vec{\boldsymbol{\nabla}}_\perp B$, we used the fact that $B \equiv |\vec{\boldsymbol{B}}| = M/r^3$ and

$$\vec{\boldsymbol{\nabla}}_\perp = \hat{\boldsymbol{n}} \cdot \vec{\boldsymbol{\nabla}} = \hat{\boldsymbol{r}}\,\frac{\partial}{\partial r} + \hat{\boldsymbol{\phi}}\,\frac{1}{r\sin\theta}\,\frac{\partial}{\partial\phi}\,,$$

where $\hat{\boldsymbol{n}} \cdot \vec{\boldsymbol{B}} = 0$. Inserting the results of eq. (67) into eq. (66), we end up with

$$\vec{\boldsymbol{v}}_G = \frac{\omega_B a^2}{2}\left(\frac{R^6}{M^2}\right)\left(\frac{M}{R^3}\right)\left(\frac{3M}{R^4}\right)\hat{\boldsymbol{\theta}} \times \hat{\boldsymbol{r}} = -\frac{3\omega_B a^2}{2R}\,\hat{\boldsymbol{\phi}}\,. \tag{68}$$

Finally, we can express $\vec{\boldsymbol{v}}_G$ in terms of the angular velocity $d\phi/dt$ by

$$\vec{\boldsymbol{v}}_G = R\,\frac{d\phi}{dt}\,\hat{\boldsymbol{\phi}}\,.$$

Comparing this equation with eq. (68), we conclude that

$$\frac{d\phi}{dt} = -\frac{3a^2}{2R^2}\,\omega_B\,.$$

Solving this differential equation, and imposing the initial condition $\phi(t_0) = \phi_0$, we end up with

$$\phi(t) = \phi_0 - \frac{3a^2}{2R^2}\,\omega_B(t-t_0)\,. \tag{69}$$

(c) If, in addition to its circular motion of part (b), the particle has a small component of velocity parallel to the lines of force, show that it undergoes small oscillations in θ around $\theta = \frac{1}{2}\pi$ with frequency $\Omega = (3/\sqrt{2})(a/R)\omega_B$. Find the change in longitude per cycle of oscillation in latitude.

As discussed in Chapter 12, section 4 of Jackson, the transverse velocity of gyration is $v_\perp = \omega_B a$ [cf. discussion below eq. (12.61) of Jackson]. If in addition, we now include the small component of the velocity parallel to the lines of magnetic force, we may use eq. (12.72) of Jackson to write:

$$v_\parallel^2 = v_0^2 - v_{\perp\,0}^2\,\frac{B(z)}{B_0}\,.$$

Here, the subscript 0 refers to the equator $z=0$ (or equivalently to $\theta = \frac{1}{2}\pi$). In particular, we can write $v_0^2 = v_{\parallel\,0}^2 + v_{\perp\,0}^2$ so that

$$v_\parallel^2 = v_{\parallel\,0}^2 + v_{\perp\,0}^2\left(1 - \frac{B(z)}{B_0}\right)\,. \tag{70}$$

In part (a), we found that along the lines of magnetic force,

$$B(\theta) = \frac{M}{r_0^3}\,\frac{\sqrt{1+3\cos^2\theta}}{\sin^6\theta}\,, \tag{71}$$

where $r_0 \equiv r(\theta = \frac{1}{2}\pi)$. In this problem, we are interested in the behavior of the particle at the mean radius R, so we take $r_0 = R$. To compute $B(z)$, we expand about $z=0$. Since $z = R\cos\theta$, we expand about $z=0$ by writing $\theta = \frac{1}{2}\pi + \epsilon$. Then,

$$z = R\cos\theta = R\cos\left(\tfrac{1}{2}\pi + \epsilon\right) = -R\sin\epsilon \simeq -R\epsilon\,.$$

Hence, $\epsilon \simeq -z/R$ and $\theta \simeq \frac{1}{2}\pi - z/R$. It follows that

$$\cos\theta \simeq \cos\left(\frac{\pi}{2} - \frac{z}{R}\right) = \sin\frac{z}{R}\,, \qquad \sin\theta \simeq \sin\left(\frac{\pi}{2} - \frac{z}{R}\right) = \cos\frac{z}{R}\,.$$

Using eq. (71),

$$B(z) \simeq \frac{M}{r_0^3}\,\frac{\sqrt{1+3\sin^2(z/R)}}{\cos^6(z/R)} \simeq \frac{M}{r_0^3}\,\frac{\sqrt{1+3z^2/R^2}}{[1-z^2/(2R^2)]^6} \simeq \frac{M}{r_0^3}\left[1 + \frac{9z^2}{2R^2}\right]\,.$$

Plugging this result into eq. (70) yields

$$v_\parallel^2(z) = v_{\parallel 0}^2 - \frac{9}{2}\left(\frac{\omega_B a}{R}\right)^2 z^2 \,. \tag{72}$$

As discussed below eq. (12.72) of Jackson, this equation is equivalent to the conservation of energy of a one-dimensional non-relativistic mechanics problem with total mechanical energy,

$$E(z) = \tfrac{1}{2} m v_\parallel^2 + V(z)\,,$$

where

$$V(z) = \tfrac{1}{2} m \left(\frac{9\omega_B^2 a^2}{2R^2}\right) z^2\,, \tag{73}$$

is the potential energy of a one-dimensional harmonic oscillator. Indeed, eq. (72) is equivalent to the statement that $E(z) = E(0)$, i.e. conservation of energy. If we write the harmonic oscillator potential in the standard form,

$$V(z) = \tfrac{1}{2} m \Omega^2 z^2\,,$$

the eq. (73) implies that the effective oscillator frequency Ω is given by

$$\Omega = \frac{3}{\sqrt{2}} \frac{\omega_B a}{R}\,.$$

That is, the charged particle undergoes small oscillations in θ around $\theta = \frac{1}{2}\pi$ with frequency Ω.

One period T of oscillation is given by

$$T = \frac{2\pi}{\Omega} = \frac{2\sqrt{2}\pi R}{3\omega_B a}\,. \tag{74}$$

Using the results of part (b) [cf. eq. (69)], the change of longitude is

$$\Delta\phi = -\frac{3a^2}{2R^2}\,\omega_B \Delta t\,. \tag{75}$$

Choosing $\Delta t = T$ then yields the change of longitude per cycle of oscillation in latitude,

$$\Delta\phi = -\frac{\sqrt{2}\pi a}{R}\,.$$

(d) For an electron of 10 MeV kinetic energy at a mean radius of $R = 3 \times 10^7$ m, find ω and a, and so determine how long it takes to drift once around the earth and how long it takes to execute one cycle of oscillation in latitude. Calculate the same quantities for an electron of 10 keV at the same radius.

Given $M = 8.1 \times 10^{25}$ gauss-cm^3 and $R = 3 \times 10^9$ cm, the magnetic field at the equator is

$$B = \frac{M}{R^3} = 3 \times 10^{-3}\ \text{gauss}\,.$$

Using eq. (12.39) of Jackson,

$$\omega_B = \frac{eB}{\gamma mc} = \frac{ecB}{\gamma mc^2}\,. \tag{76}$$

Although the last step above is rather trivial, it is convenient to write ω_B in this form. The numerical value of the quantity ec is given by

$$ec = (4.8 \times 10^{-10}\ \text{statcoulombs})(3 \times 10^{10}\ \text{cm s}^{-1}) = 14.4\ \text{statcoulombs cm s}^{-1}\,. \tag{77}$$

It is convenient to eliminate statcoulombs in favor of gauss. That is,

$$1\ \text{gauss} = 1\ \text{dyne statcoulomb}^{-1} = 1\ \text{erg cm}^{-1}\ \text{statcoulomb}^{-1}\,.$$

Using $1\ \text{eV} = 1.6 \times 10^{-12}$ ergs, we can write:

$$1\ \text{gauss} = (1.6 \times 10^{-12})^{-1}\ \text{eV cm}^{-1}\ \text{statcoulomb}^{-1} = 6.25 \times 10^{11}\ \text{eV cm}^{-1}\ \text{statcoulomb}^{-1}\,.$$

Hence, it follows that

$$1\ \text{statcoulomb} = 6.25 \times 10^{11}\ \text{eV cm}^{-1}\ \text{gauss}^{-1}\,.$$

Inserting this result into eq. (77) yields

$$ec = 9 \times 10^{12}\ \text{eV gauss}^{-1}\ \text{s}^{-1}\,.$$

Therefore, the gyration frequency can be written as

$$\omega_B = 9 \times 10^{12}\ \text{s}^{-1}\,\frac{B\ (\text{gauss})}{\gamma mc^2\,(\text{eV})}\,. \tag{78}$$

For the electron, $mc^2 = 511$ keV. If the electron has a kinetic energy of $K = 10$ MeV, then $E = \gamma mc^2 = mc^2 + K$, which yields $K = (\gamma - 1)mc^2$. Hence,

$$\gamma = 1 + \frac{K}{mc^2} = 1 + \frac{10\ \text{MeV}}{0.511\ \text{MeV}} = 20.57\,.$$

It follows from eq. (78) that

$$\omega_B = 9 \times 10^{12}\ \text{s}^{-1} \cdot \frac{3 \times 10^{-3}}{(20.57)(5.11 \times 10^5)} = 2.57 \times 10^3\ \text{s}^{-1}\,.$$

Next we use $v \simeq v_\perp = \omega_B a$ to determine a. Since $\gamma \gg 1$, it follows that $v \simeq c$, so that

$$a = \frac{c}{\omega_B} = \frac{3 \times 10^{10}\ \text{cm s}^{-1}}{2.57 \times 10^3\ \text{s}^{-1}} = 117\ \text{km}\,.$$

To drift once around the earth requires the longitude (or azimuthal angle ϕ) to change by 2π. Inserting $\Delta\phi = -2\pi$ in eq. (75) [the overall sign is not significant here], we obtain

$$\Delta t = \frac{4\pi R^2}{3a^2\omega_B} = \frac{4\pi(3 \times 10^9\ \text{cm})^2}{3(1.17 \times 10^7\ \text{cm})^2(2.57 \times 10^3\ \text{s}^{-1})} = 107\ \text{s}\,.$$

Finally, the time it takes to execute one cycle of oscillation in latitude was obtained in part (c) [cf. eq. (74)]:

$$T = \frac{2\sqrt{2}\pi R}{3\omega_B a} = \frac{2\sqrt{2}\pi(3\times 10^9\ \text{cm})}{3(1.17\times 10^7\ \text{cm})^2(2.57\times 10^3\ \text{s}^{-1})} = 0.3\ \text{s}\,.$$

For an electron with kinetic energy of 10 keV,

$$\gamma = 1 + \frac{K}{mc^2} = 1 + \frac{10\ \text{keV}}{511\ \text{keV}} = 1.02\,. \tag{79}$$

It follows from eq. (78) that

$$\omega_B = \frac{(9\times 10^{12}\ \text{s}^{-1})(3\times 10^{-3})}{(1.02)(5.11\times 10^5)} = 5.18\times 10^4\ \text{s}^{-1}\,.$$

To determine a, we first compute v using eq. (79):

$$\frac{1}{\sqrt{1-v^2/c^2}} = 1.02 \quad\Longrightarrow\quad \frac{v}{c} = 0.195\,.$$

Hence,

$$a = \frac{v}{\omega_B} = \frac{(0.195)(3\times 10^{10}\ \text{cm s}^{-1})}{5.18\times 10^4\ \text{s}^{-1}} = 1.13\ \text{km}\,.$$

Finally, following the previous computation,

$$\Delta t = \frac{4\pi R^2}{3a^2\omega_B} = \frac{4\pi(3\times 10^9\ \text{cm})^2}{3(1.13\times 10^5\ \text{cm})^2(5.18\times 10^4\ \text{s}^{-1})} = 5.7\times 10^4\ \text{s}\,,$$

and

$$T = \frac{2\sqrt{2}\pi R}{3\omega_B a} = \frac{2\sqrt{2}\pi(3\times 10^9\ \text{cm})}{3(1.13\times 10^5\ \text{cm})^2(5.18\times 10^4\ \text{s}^{-1})} = 1.52\ \text{s}\,.$$

Note that in both computations above, we have $a \ll R$, which implies that the gradient of the magnetic field is small over the orbit of the electrons. Hence, the approximations introduced in Chapter 12, sections 4 and 5 of Jackson are valid for the charged particle motions examined in this problem.

5. [Jackson, problem 12.11] Consider the precession of the spin of a muon, initially longitudinally polarized, as the muon moves in a circular orbit in a plane perpendicular to a uniform magnetic field $\vec{\boldsymbol{B}}$.

(a) Show that the difference Ω of the spin precession frequency and the orbital gyration frequency is

$$\Omega = \frac{eBa}{m_\mu c}\,,$$

independent of the muon's energy, where $a = \frac{1}{2}(g-2)$ is the magnetic moment anomaly. Find the equations of motion for the components of the spin along the mutually perpendicular directions defined by the particle's velocity, the radius vector from the center of the circle to the particle, and the magnetic field.

Our starting point is the Thomas equation, which Jackson writes in the following form [cf. eq. (11.170) of Jackson]:

$$\frac{d\vec{\boldsymbol{s}}}{dt} = \frac{e}{mc}\,\vec{\boldsymbol{s}}\times\left\{\left(\frac{g}{2}-1+\frac{1}{\gamma}\right)\vec{\boldsymbol{B}} - \left(\frac{g}{2}-1\right)\frac{\gamma}{\gamma+1}(\vec{\boldsymbol{\beta}}\boldsymbol{\cdot}\vec{\boldsymbol{B}})\vec{\boldsymbol{\beta}} - \left(\frac{g}{2}-\frac{\gamma}{\gamma+1}\right)\vec{\boldsymbol{\beta}}\times\vec{\boldsymbol{E}}\right\}\,, \tag{80}$$

where the time derivative of the velocity vector is given by [cf. eq. (11.168) of Jackson]:

$$\frac{d\vec{\boldsymbol{\beta}}}{dt} = \frac{e}{\gamma mc}\left[\vec{\boldsymbol{E}}+\vec{\boldsymbol{\beta}}\times\vec{\boldsymbol{B}}-\vec{\boldsymbol{\beta}}(\vec{\boldsymbol{\beta}}\boldsymbol{\cdot}\vec{\boldsymbol{E}})\right]\,. \tag{81}$$

For a particle moving in a circular orbit in a plane perpendicular to a uniform magnetic field $\vec{\boldsymbol{B}}$, we have $\vec{\boldsymbol{\beta}}\boldsymbol{\cdot}\vec{\boldsymbol{B}}=0$, where $\vec{\boldsymbol{v}}\equiv c\vec{\boldsymbol{\beta}}$ is the particle velocity. Hence, eqs. (80) and (81) reduce to

$$\frac{d\vec{\boldsymbol{s}}}{dt} = \frac{e}{mc}\left(\frac{g}{2}-1+\frac{1}{\gamma}\right)\vec{\boldsymbol{s}}\times\vec{\boldsymbol{B}}\,,\qquad \frac{d\vec{\boldsymbol{v}}}{dt} = \frac{e}{\gamma mc}\,\vec{\boldsymbol{v}}\times\vec{\boldsymbol{B}}\,, \tag{82}$$

since by assumption there is no electric field present ($\vec{\boldsymbol{E}}=0$). That is, eq. (82) can be written in the form of precession equations,

$$\frac{d\vec{\boldsymbol{s}}}{dt} = \vec{\boldsymbol{s}}\times\vec{\boldsymbol{\omega}}\,,\qquad \frac{d\vec{\boldsymbol{v}}}{dt} = \vec{\boldsymbol{v}}\times\vec{\boldsymbol{\omega}}_B\,,$$

where the spin precession frequency $\vec{\boldsymbol{\omega}}$ and the orbital gyration frequency $\vec{\boldsymbol{\omega}}_B$ are given by:

$$\vec{\boldsymbol{\omega}} \equiv \frac{e}{\gamma mc}\left[1+\left(\frac{g-2}{2}\right)\gamma\right]\vec{\boldsymbol{B}}\,,\qquad \vec{\boldsymbol{\omega}}_B \equiv \frac{e}{\gamma mc}\vec{\boldsymbol{B}}\,.$$

The difference of these two frequencies is

$$\vec{\boldsymbol{\Omega}} \equiv \vec{\boldsymbol{\omega}} - \vec{\boldsymbol{\omega}}_B = \frac{e}{mc}\left(\frac{g-2}{2}\right)\vec{\boldsymbol{B}}\,,$$

and the magnitude of this frequency difference is given by

$$\Omega = \frac{eBa}{mc}\,,\qquad \text{where } a = \tfrac{1}{2}(g-2)\,.$$

To find the equations of motion for the components of the spin vector, we first decompose this vector into longitudinal and transverse components with respect to the direction of the velocity, $\hat{\boldsymbol{\beta}}\equiv\vec{\boldsymbol{\beta}}/\beta$. That is, $\vec{\boldsymbol{s}}=\vec{\boldsymbol{s}}_\parallel+\vec{\boldsymbol{s}}_\perp$, where

$$\vec{\boldsymbol{s}}_\parallel = (\hat{\boldsymbol{\beta}}\boldsymbol{\cdot}\vec{\boldsymbol{s}})\hat{\boldsymbol{\beta}}\,,\qquad \vec{\boldsymbol{s}}_\perp = \vec{\boldsymbol{s}}-\vec{\boldsymbol{s}}_\parallel\,.$$

By construction,

$$\vec{\boldsymbol{s}}_\perp\boldsymbol{\cdot}\hat{\boldsymbol{\beta}} = 0\,. \tag{83}$$

We first work out $d\vec{\boldsymbol{s}}_\parallel/dt$.

$$\frac{d\vec{\boldsymbol{s}}_\parallel}{dt} = \frac{d}{dt}\Big((\hat{\boldsymbol{\beta}}\cdot\vec{\boldsymbol{s}})\hat{\boldsymbol{\beta}}\Big) = \hat{\boldsymbol{\beta}}\,\frac{d}{dt}\Big(\hat{\boldsymbol{\beta}}\cdot\vec{\boldsymbol{s}}\Big) + \vec{\boldsymbol{s}}\cdot\hat{\boldsymbol{\beta}}\,\frac{d\hat{\boldsymbol{\beta}}}{dt}\,. \tag{84}$$

Jackson gives the following result in his eq. (11.171),

$$\frac{d}{dt}\Big(\hat{\boldsymbol{\beta}}\cdot\vec{\boldsymbol{s}}\Big) = -\frac{e}{mc}\vec{\boldsymbol{s}}_\perp\cdot\left[\left(\frac{g}{2}-1\right)\hat{\boldsymbol{\beta}}\times\vec{\boldsymbol{B}} + \left(\frac{g\beta}{2}-\frac{1}{\beta}\right)\vec{\boldsymbol{E}}\right]\,.$$

Setting $\vec{\boldsymbol{E}}=0$, we obtain

$$\frac{d}{dt}\Big(\hat{\boldsymbol{\beta}}\cdot\vec{\boldsymbol{s}}\Big) = -\frac{eB}{mc}\left(\frac{g-2}{2}\right)\vec{\boldsymbol{s}}_\perp\cdot(\hat{\boldsymbol{\beta}}\times\hat{\boldsymbol{B}})\,. \tag{85}$$

We also need to work out $d\hat{\boldsymbol{\beta}}/dt$.

$$\frac{d\hat{\boldsymbol{\beta}}}{dt} = \frac{d}{dt}\left(\frac{\vec{\boldsymbol{\beta}}}{\beta}\right) = \frac{1}{\beta}\,\frac{d\vec{\boldsymbol{\beta}}}{dt} - \frac{\vec{\boldsymbol{\beta}}}{\beta^2}\,\frac{d\beta}{dt}\,. \tag{86}$$

Using

$$\frac{d\beta}{dt} = \frac{d}{dt}\Big(\vec{\boldsymbol{\beta}}\cdot\vec{\boldsymbol{\beta}}\Big)^{1/2} = \frac{1}{2}\Big(\vec{\boldsymbol{\beta}}\cdot\vec{\boldsymbol{\beta}}\Big)^{-1/2}\frac{d}{dt}\Big(\vec{\boldsymbol{\beta}}\cdot\vec{\boldsymbol{\beta}}\Big) = \frac{1}{2\beta}\,2\vec{\boldsymbol{\beta}}\cdot\frac{d\vec{\boldsymbol{\beta}}}{dt} = \hat{\boldsymbol{\beta}}\cdot\frac{d\vec{\boldsymbol{\beta}}}{dt}\,,$$

in eq. (86), we conclude that

$$\frac{d\hat{\boldsymbol{\beta}}}{dt} = \frac{1}{\beta}\left[\frac{d\vec{\boldsymbol{\beta}}}{dt} - \hat{\boldsymbol{\beta}}\left(\hat{\boldsymbol{\beta}}\cdot\frac{d\vec{\boldsymbol{\beta}}}{dt}\right)\right]\,.$$

From eq. (82), we obtain

$$\frac{d\vec{\boldsymbol{\beta}}}{dt} = \frac{e}{\gamma mc}\,\vec{\boldsymbol{\beta}}\times\vec{\boldsymbol{B}}\,.$$

Hence $\hat{\boldsymbol{\beta}}\cdot d\vec{\boldsymbol{\beta}}/dt = 0$, and we end up with

$$\frac{d\hat{\boldsymbol{\beta}}}{dt} = \frac{eB}{\gamma mc}\,\hat{\boldsymbol{\beta}}\times\hat{\boldsymbol{B}}\,. \tag{87}$$

Inserting eqs. (85) and (87) into eq. (84), we obtain

$$\frac{ds_\parallel}{dt} = -\frac{eB}{mc}\left(\frac{g-2}{2}\right)[\vec{\boldsymbol{s}}_\perp\cdot(\hat{\boldsymbol{\beta}}\times\hat{\boldsymbol{B}})]\hat{\boldsymbol{\beta}} + \frac{eB}{\gamma mc}\,\vec{\boldsymbol{s}}\cdot\hat{\boldsymbol{\beta}}(\hat{\boldsymbol{\beta}}\times\hat{\boldsymbol{B}})\,.$$

Since $\vec{\boldsymbol{s}}_\parallel \equiv (\vec{\boldsymbol{s}}\cdot\hat{\boldsymbol{\beta}})\hat{\boldsymbol{\beta}}$, it immediately follows that

$$\vec{\boldsymbol{s}}\cdot\hat{\boldsymbol{\beta}}(\hat{\boldsymbol{\beta}}\times\hat{\boldsymbol{B}}) = \vec{\boldsymbol{s}}_\parallel\times\vec{\boldsymbol{B}}\,.$$

We can further simplify the quantity $[\vec{s}_\perp \cdot (\hat{\beta} \times \hat{B})]\hat{\beta}$ by using $\vec{s}_\perp \cdot \hat{\beta} = 0$ [cf. eq. (83)] and $\hat{\beta} \cdot \hat{B} = 0$. First, consider the triple cross product

$$\vec{s}_\perp \times \left[\hat{\beta} \times (\hat{\beta} \times \hat{B})\right] = [\vec{s}_\perp \cdot (\hat{\beta} \times \hat{B})]\hat{\beta} - (\hat{\beta} \times \hat{B})\vec{s}_\perp \cdot \hat{\beta} = [\vec{s}_\perp \cdot (\hat{\beta} \times \hat{B})]\hat{\beta}\,.$$

However, $\hat{\beta} \times (\hat{\beta} \times \hat{B}) = \hat{\beta}(\hat{\beta} \cdot \hat{B}) - \hat{B} = -\hat{B}$. Hence,

$$[\vec{s}_\perp \cdot (\hat{\beta} \times \hat{B})]\hat{\beta} = -\vec{s}_\perp \times \hat{B}\,.$$

Inserting eqs. (89) and (90) into eq. (88) then yields

$$\boxed{\frac{d\vec{s}_\parallel}{dt} = \frac{eB}{mc}\left[\left(\frac{g-2}{2}\right)\vec{s}_\perp + \frac{1}{\gamma}\,\vec{s}_\parallel\right] \times \hat{B}}$$

Using this result, we can evaluate $d\vec{s}_\perp/dt$.

$$\frac{d\vec{s}_\perp}{dt} = \frac{d}{dt}\left(\vec{s} - \vec{s}_\parallel\right) = \frac{eB}{mc}\left(\frac{g}{2} - 1 + \frac{1}{\gamma}\right)(\vec{s}_\parallel + \vec{s}_\perp) \times \vec{B} = \frac{eB}{mc}\left[\left(\frac{g-2}{2}\right)\vec{s}_\perp + \frac{1}{\gamma}\,\vec{s}_\parallel\right] \times \hat{B}\,,$$

which simplifies to

$$\boxed{\frac{d\vec{s}_\perp}{dt} = \frac{eB}{mc}\left[\left(\frac{g-2}{2}\right)\vec{s}_\parallel + \frac{1}{\gamma}\,\vec{s}_\perp\right] \times \hat{B}}$$

Finally, we need to further decompose $\vec{s}_\perp$ into components along the direction of the magnetic field and along the direction of the unit radius vector $\hat{r}$ that points to the center of the circular path of the moving spin. In light of eq. (81) [with $\vec{E} = 0$], $d\vec{v}/dt \propto \hat{\beta} \times \hat{B}$, where $\hat{\beta} \cdot \hat{B} = 0$. But for circular motion, $\hat{r} \cdot \hat{\beta} = 0$ and the acceleration $d\vec{v}/dt$ points radially into the origin, i.e. $d\vec{v}/dt \propto -\hat{r}$. It follows that $\hat{r} = \hat{B} \times \hat{\beta}$, and we conclude that the unit vectors $\{\hat{B}\,, \hat{\beta}\,, \hat{r}\}$ form a mutually orthonormal right-handed triad of vectors. Thus, we can write:

$$\vec{s}_\perp \equiv \vec{s}_B + \vec{s}_r\,, \qquad \text{where} \ \ \vec{s}_B \equiv (\vec{s} \cdot \hat{B})\hat{B} \text{ and } \vec{s}_r \equiv (\vec{s} \cdot \hat{r})\hat{r}\,. \tag{88}$$

Note that

$$\frac{d\vec{s}_B}{dt} = \left(\hat{B} \cdot \frac{d\vec{s}}{dt}\right)\hat{B} = 0\,, \tag{89}$$

since $\vec{B}$ is time-independent by assumption and

$$\vec{B} \cdot \frac{d\vec{s}}{dt} \propto \vec{B} \cdot (\vec{s} \times \vec{B}) = 0\,,$$

in light of eq. (82). Thus, $\vec{s}_B$ is a constant in time, from which it follows that

$$\frac{d\vec{s}_r}{dt} = \frac{d}{dt}\,(\vec{s}_\perp + \vec{s}_B) = \frac{d\vec{s}_\perp}{dt}\,. \tag{90}$$

Hence, the equations of motion for the components of the spin vector are:

$$\frac{d\vec{\boldsymbol{s}}_B}{dt} = 0\,,$$

$$\frac{d\vec{\boldsymbol{s}}_r}{dt} = \frac{eB}{mc}\left[\left(\frac{g-2}{2}\right)\vec{\boldsymbol{s}}_{\parallel} + \frac{1}{\gamma}\,\vec{\boldsymbol{s}}_r\right] \times \hat{\boldsymbol{B}}\,,$$

$$\frac{d\vec{\boldsymbol{s}}_{\parallel}}{dt} = \frac{eB}{mc}\left[\left(\frac{g-2}{2}\right)\vec{\boldsymbol{s}}_r + \frac{1}{\gamma}\,\vec{\boldsymbol{s}}_{\parallel}\right] \times \hat{\boldsymbol{B}}\,,$$

after using $\vec{\boldsymbol{s}}_B \times \hat{\boldsymbol{B}} = (\vec{\boldsymbol{s}} \cdot \hat{\boldsymbol{B}})\hat{\boldsymbol{B}} \times \hat{\boldsymbol{B}} = 0$.

(b) For the CERN Muon Storage Ring, the orbit radius is $R = 2.5$ meters and $B = 17 \times 10^3$ gauss. What is the momentum of the muon? What is the time dilation factor γ? How many periods of precession $T = 2\pi/\Omega$ occur per observed laboratory mean lifetime of the muons? [Relevant data: $m_\mu = 105.66$ MeV, $\tau_0 = 2.2 \times 10^{-6}$ s, $a \simeq \alpha/(2\pi)$ where $\alpha \simeq 1/137$.]

For circular motion,

$$\vec{\boldsymbol{a}} = \frac{d\vec{\boldsymbol{v}}}{dt} = -\frac{v^2}{R}\,\hat{\boldsymbol{r}}\,. \tag{91}$$

Since the circular motion is in a plane that is perpendicular to the magnetic field $\vec{\boldsymbol{B}}$, it follows that $\vec{\boldsymbol{B}}$, $\vec{\boldsymbol{v}}$ and $\hat{\boldsymbol{r}}$ are mutually orthogonal vectors. Moreover, eqs. (12.38) and (12.39) of Jackson yield

$$\frac{d\vec{\boldsymbol{v}}}{dt} = \frac{e}{\gamma mc}\vec{\boldsymbol{v}} \times \vec{\boldsymbol{B}}\,. \tag{92}$$

Thus, if $\vec{\boldsymbol{B}}$ points in the z-direction, then $\vec{\boldsymbol{v}} = -v\hat{\boldsymbol{\theta}}$ and the circular motion is clockwise in the x–y plane. Combining eqs. (91) and (92), it follows that

$$\gamma mv = \frac{eBR}{c}\,, \tag{93}$$

which we recognize as the relativistic momentum of the muon, p_μ. Using eq. (12.42) of Jackson, we can rewrite eq. (93) as

$$p_\mu\ (\text{MeV/c}) = 3 \times 10^{-4}\ BR\ (\text{gauss-cm})\,. \tag{94}$$

The factor of 3×10^{-4} in eq. (94) arises as follows. In gaussian units, $e = 4.8 \times 10^{-10}$ esu and 1 MeV$= 1.6 \times 10^{-6}$ ergs. Hence, the conversion factor between ergs and MeV is

$$4.8 \times 10^{-10}/1.6 \times 10^{-6} = 3 \times 10^{-4}.$$

Thus we end up with

$$p_\mu = (3 \times 10^{-4})(1.7 \times 10^4)(250)\ \text{MeV/c} = 1.275 \times 10^3\ \text{MeV/c}\,.$$

The γ-factor is

$$\gamma = \frac{E}{mc^2} = \frac{(p^2c^2 + m^2c^4)^{1/2}}{mc^2} = \left(\frac{p^2}{m^2c^2} + 1\right)^{1/2} .$$

The muon rest energy is $mc^2 = 105.66$ MeV. Hence,

$$\gamma = \left[1 + \frac{(1.275 \times 10^3)^2}{(105.66)^2}\right]^{1/2} = 12.11 .$$

The number of periods of precession, $T = 2\pi/\Omega$, occurring per observed mean muon lifetime, $\gamma\tau_0 = \gamma(2.2 \times 10^{-6}\text{ s})$, is given by[8]

$$\frac{\gamma\tau_0}{T} = \frac{\gamma\tau_0\Omega}{2\pi} = \frac{\gamma\tau_0 eBa}{2\pi mc} = \frac{\gamma^2\tau_0 va}{2\pi R} ,$$

where eq. (93) was used to arrive at the final result above. Since $\gamma \gg 1$, we can approximate $v \simeq c$. In addition, we take

$$a = \tfrac{1}{2}(g-2) \simeq \frac{\alpha}{2\pi} , \qquad \text{where } \alpha \simeq \frac{1}{137} ,$$

as predicted at lowest non-trivial order in quantum electrodynamics. Hence,

$$\frac{\gamma\tau_0}{T} \simeq \frac{\gamma^2\tau_0 c\alpha}{4\pi^2 R} = \frac{(12.11)^2(2.2 \times 10^{-6}\text{ s})(3 \times 10^{10}\text{ cm s}^{-1})}{4\pi^2(250\text{ cm})(137)} = 7.156 .$$

(c) Express the difference frequency Ω in units of orbital rotation frequency and compute how many precessional periods (at the difference frequency) occur per rotation for a 300 MeV muon, a 300 MeV electron, a 5 GeV electron (this last typical of the e^+e^- storage ring at Cornell).

NOTE: The energy values above correspond to the total relativistic energies.

For a 300 MeV muon,

$$\gamma = \frac{E}{mc^2} = \frac{300}{105.66} = 2.839 ,$$

and

$$\Omega = \frac{eBa}{mc} = \gamma\omega_B a \simeq \frac{\gamma\omega_B\alpha}{2\pi} = 3.3 \times 10^{-3}\omega_B .$$

One revolution occurs in time $t = 2\pi R/v$. In this time, the number of periods of precession, $T = 2\pi/\Omega$, is given by

$$\frac{t}{T} = \left(\frac{2\pi R}{v}\right)\left(\frac{\Omega}{2\pi}\right) = \frac{\Omega R}{v} .$$

[8]Note that in the laboratory frame, the observed muon lifetime is given by $\gamma\tau_0$, where τ_0 is the muon lifetime in the muon rest frame.

We can rewrite the above result using eq. (93), which yields

$$\frac{R}{v} = \frac{\gamma mc}{eB} = \frac{1}{\omega_B}\,.$$

Hence, for a 300 MeV muon, we have

$$\frac{t}{T} = \frac{\Omega}{\omega_B} \simeq \frac{\gamma\alpha}{2\pi} = 3.3 \times 10^{-3}\,.$$

For a 300 MeV electron, we use $m_e c^2 = 511$ keV to obtain $\gamma = 300/0.511 = 587$. Hence,

$$\frac{t}{T} = \frac{\Omega}{\omega_B} \simeq \frac{\gamma\alpha}{2\pi} = 0.682\,.$$

Finally, for a 5 GeV electron, we have $\gamma = 500/0.511 = 9.785 \times 10^3$. It follows that

$$\frac{t}{T} = \frac{\Omega}{\omega_B} \simeq \frac{\gamma\alpha}{2\pi} = 11.37\,.$$