This is an open book exam with a time limit of three hours. You may refer to Jackson's text, the class handouts and solution sets, or any other material linked to the course webpage. You may also consult your own personal notes and any reference of integrals or other mathematical facts. However, you may not collaborate with anyone else during the exam.

Note that you do not have to derive all results from scratch. However, if you use a particular result, you should cite its source (e.g. an equation in Jackson, an equation derived in a class lecture, or an equation that appears in a solution set or in a class handout).

This exam consists of four problems consisting of ten individual parts, each of which is worth ten points. (Two additional parts marked "extra credit" are optional.) Use this information to manage your time appropriately during the exam. In addition, I have included three opportunities (worth 10 points each) for extra credit.

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1. [40] Consider an oversimplified model of an antenna consisting of a thin wire of length ℓ and negligible cross section, carrying a harmonically varying current density flowing in the z direction. The (complex) current in the wire is given by $Ie^{-i\omega t}$, where I is a constant (independent of position).

(a) Show that the (complex) current density takes the form:

$$\vec{\boldsymbol{J}}(\vec{\boldsymbol{x}}, t) = \hat{\boldsymbol{z}} I e^{-i\omega t}\delta(x)\delta(y)[\Theta(z + \tfrac{1}{2}\ell) - \Theta(z - \tfrac{1}{2}\ell)]\,, \tag{1}$$

by verifying that eq. (1) implies that the corresponding current is given by $Ie^{-i\omega t}$ where the step function $\Theta(x) \equiv 1$ if $x > 0$ and $\Theta(x) \equiv 0$ if $x < 0$. Here, we have assumed that the point $z = 0$ corresponds to the midpoint of the antenna.

(b) Prove that there is an oscillating charge density at $z = \pm\frac{1}{2}\ell$ (i.e., at both ends of the antenna), but the charge density vanishes at any interior point on the antenna.

(c) Show that the antenna acts like an oscillating electric dipole moment, $\vec{\boldsymbol{p}}\, e^{-i\omega t}$. Evaluate $\vec{\boldsymbol{p}}$ in terms of the current I, the antenna length ℓ and the angular frequency ω.

(d) Calculate the angular distribution of the radiated power, $dP/d\Omega$, assuming that $\lambda \gg \ell$, where λ is the wavelength of the emitted radiation. Express your answer in terms of the current I, the antenna length ℓ and the wavelength λ. Integrate over angles to obtain the total radiated power.

2. [20] Consider a relativistic particle with charge e moving along a trajectory $\vec{\boldsymbol{r}}(t)$ at velocity $\vec{\boldsymbol{v}} = c\vec{\boldsymbol{\beta}}(t) \equiv d\vec{\boldsymbol{r}}(t)/dt$ and acceleration $\vec{\boldsymbol{a}}(t) \equiv d\vec{\boldsymbol{v}}(t)/dt$. In addition to radiating energy, the particle also radiates angular momentum. Denote the angular momentum radiated per unit time by $\vec{\boldsymbol{\tau}} \equiv d\vec{\boldsymbol{L}}/dt$.

(a) Show that the angular distribution of the angular momentum radiated per unit *retarded* time in gaussian units is given by

$$\frac{d\vec{\boldsymbol{\tau}}'}{d\Omega} = \frac{e^2(1-\beta^2)}{4\pi c}\left[\frac{(1-\vec{\boldsymbol{\beta}}\boldsymbol{\cdot}\hat{\boldsymbol{n}})\left(\hat{\boldsymbol{n}}\times\dot{\vec{\boldsymbol{\beta}}}\right)+(\hat{\boldsymbol{n}}\times\vec{\boldsymbol{\beta}})\hat{\boldsymbol{n}}\boldsymbol{\cdot}\dot{\vec{\boldsymbol{\beta}}}}{(1-\vec{\boldsymbol{\beta}}\boldsymbol{\cdot}\hat{\boldsymbol{n}})^4}\right],$$

where $\vec{\boldsymbol{\tau}}' \equiv d\vec{\boldsymbol{L}}/dt_{\rm ret}$ and $\dot{\vec{\boldsymbol{\beta}}} \equiv d\vec{\boldsymbol{\beta}}/dt_{\rm ret}$.

(b) Using the result of part (a), integrate over angles to obtain the total angular momentum radiated per unit retarded time.

<u>*HINT:*</u> The following integrals will be useful:

$$\int \frac{\hat{\boldsymbol{n}}\,d\Omega}{(1-\vec{\boldsymbol{\beta}}\boldsymbol{\cdot}\hat{\boldsymbol{n}})^3} = \frac{4\pi\vec{\boldsymbol{\beta}}}{(1-\beta^2)^2}\,, \tag{2}$$

$$\int \frac{\hat{n}_i\hat{n}_j\,d\Omega}{(1-\vec{\boldsymbol{\beta}}\boldsymbol{\cdot}\hat{\boldsymbol{n}})^4} = \frac{4\pi}{3(1-\beta^2)^2}\left[\delta_{ij}+\frac{4\beta_i\beta_j}{1-\beta^2}\right]. \tag{3}$$

Extra credit is available if you provide derivations for eqs. (2) and (3).

(c) [*EXTRA CREDIT*] The energy radiated per unit retarded time is denoted by P'. Consider the case where the particle moves in a circle of radius R. Compute $P'/|\vec{\boldsymbol{\tau}}'|$ in the nonrelativistic limit and interpret your result.

3. [20] A particle of charge e initially moves along a straight line at constant velocity v_0. Its velocity then decreases uniformly from v_0 to zero in a time interval T. At the end of this time interval, the particle remains at rest.

(a) Find the angular distribution of the radiated *energy* E emitted during the time interval of constant deceleration. Do *not* assume that v_0 is small compared to the speed of light.

(b) Suppose $v_0 \ll c$. Calculate $dE/d\Omega$ (to leading order in v_0/c). Integrate over angles to determine the total radiated energy emitted during the deceleration.

4. [20] Consider the scattering of an electromagnetic wave of (angular) frequency ω and polarization $\hat{\boldsymbol{\varepsilon}}_0$ off of an electron bound in an atom. The wavelength of the incoming wave is assumed to be significantly larger than the size of the atom. You may also assume that the non-relativistic limit is a good approximation.

One can model the electron by assuming that it is bound by a damped harmonic oscillator force with oscillation frequency ω_0 and damping coefficient η. The electron, with mass m and charge $-e$, also experiences a force due to the electric field of the incoming wave. The response of the electron to the initial wave is an induced time-dependent electric dipole moment, $\vec{\boldsymbol{p}}(t) = \mathrm{Re}(\vec{\boldsymbol{p}}\, e^{-i\omega t})$, where the complex vector $\vec{\boldsymbol{p}}$ is given by

$$\vec{\boldsymbol{p}} = \frac{e^2}{m}\left(\frac{E_0\,\hat{\boldsymbol{\epsilon}}_0}{\omega_0^2 - \omega^2 - i\eta\omega}\right) . \tag{4}$$

(a) Assuming that the incoming wave is left circularly polarized and propagates in the positive z-direction, compute the angular distribution of the scattering cross section, under the assumption that the final state polarization $\hat{\boldsymbol{\varepsilon}}$ is not observed. Express the coefficient of the angular factor as a function of the frequency of the incoming wave.

(b) Integrate the differential cross section over angles to obtain the total scattering cross section. Compare your result to the Thomson cross section in the following three limiting cases: (i) $\omega \gg \omega_0 \sim \eta$; (ii) $\omega = \omega_0 \gg \eta$; and (iii) $\omega \ll \omega_0 \sim \eta$.

(c) [*EXTRA CREDIT*] Derive eq. (4).