DUE: TUESDAY, APRIL 13, 2010

1. Prove that

$$\langle x\,,\,t\,|\,T[X(t_1)X(t_2)\cdots X(t_n)]\,|x'\,,\,t'\rangle = \int \mathcal{D}[x(t)]\,x(t_1)x(t_2)\cdots x(t_n)\,e^{iS[x(t)]/\hbar}\,,$$

where T is the time-ordered product symbol, $S[x(t)]$ is the action [which depends on the path $x(t)$], $X(t)$ is the position operator in the Heisenberg picture, and $x(t)$ is the eigenvalue of $X(t)$ when acting on the position eigenstate $|x\,,\,t\rangle$. Assume that $t \geq t_i$ and $t' \leq t_i$ for all $i = 1, 2, \ldots, n$.

2. Using the path integral technique, compute the propagator for a particle in a linear potential, where the corresponding Lagrangian is given by:

$$L(x, \dot{x}) = \tfrac{1}{2}m\dot{x}^2 + fx\,.$$

3. You are the creator of a wonderful theory that uniquely predicts the quark-antiquark potential to be:

$$V(r) = \frac{a|\vec{\boldsymbol{r}}|}{\hbar c}\,, \tag{1}$$

with $a = 0.24$ GeV^2. However, before you can claim The Prize, there is a small formality: a comparison between theory and experiment.

(a) Using the potential specified in eq. (1), calculate the energy differences,[1]

$$\Delta E_n \equiv E_{ns} - E_{1s}\,,$$

for a few lowest lying s-states ($\ell = 0$) of the bound state of a charmed quark and charmed antiquark, called charmonium ($c\bar{c}$), and the lowest lying s-states of the bound state of a bottom quark and bottom antiquark, called bottomonium ($b\bar{b}$). Do the calculation in two ways:

(i) Use the WKB approximation.

(ii) Solve the problem exactly. (A table of zeros of Airy functions and other useful facts can be found in *Handbook of Mathematical Functions*, by Abramowitz and Stegun.)

How well does the WKB approximation do?

[1] Here, n is the principal quantum number, where E_{1s} corresponds to the ground state energy.

(b) Using $m_c = m_{\bar{c}} = 1.5\ \text{GeV/c}^2$, and $m_b = m_{\bar{b}} = 4.5\ \text{GeV/c}^2$ for the masses of the charmed and bottom quarks, respectively, compare the theoretical results obtained in part (a) with the experimental results for ΔE_2, ΔE_3, ΔE_4, which are respectively given by:

(i) for charmonium: 0.590 GeV, 0.930 GeV, 1.320 GeV.

(ii) for bottomonium: 0.560 GeV, 0.890 GeV, 1.120 GeV.

(Are you going to travel to Stockholm after all?)

4. Consider the one-dimensional problem of tunneling through a smooth potential barrier, $V(x)$. Let x_1 and x_2 be the two turning points, and let

$$\kappa(x) \equiv \frac{\sqrt{2m(V(x)-E)}}{\hbar}\,, \qquad \text{for}\quad x_1 < x < x_2\,,$$

where E is the energy of the particle. Define the quantity:

$$S \equiv \exp\left\{-\int_{x_1}^{x_2} \kappa(x')\,dx'\right\}\,.$$

(a) Show that in order for the WKB approximation to be valid, we must have $S \ll 1$.

(b) Show that the transmission coefficient in the WKB approximation is given by:

$$T = \left(\frac{S}{4} + \frac{1}{S}\right)^{-2}\,.$$

Assuming that the WKB approximation is valid, we can use $S \ll 1$ to approximate the above result by

$$T \simeq S^2 = \left[\exp\left\{-\int_{x_1}^{x_2} \kappa(x')\,dx'\right\}\right]^2\,.$$

HINT: Write separate formulae for the WKB approximate wave functions in the three regions: I ($x < x_1$), II ($x_1 < x < x_2$) and III ($x > x_2$). Assume an incident wave and a reflected wave in region I, and a transmitted wave in region III. The wave function in region II will consist of a growing and a decaying exponential, so you will need to derive two new connection formulae to connect the growing exponential in region II to the corresponding wave functions in region I and region III, respectively. Once you have the relevant connection formulae, then you can write down two expressions for the wave function in region II. Setting these two expressions equal, one can deduce an expression for the transmission coefficient.

(c) In order for a charged particle of charge $Z_1 e$ to reach a nucleus of charge $Z_2 e$, it must penetrate the Coulomb barrier. Let us model this by a simple one-dimensional potential:

$$V(x) = -\frac{Z_1 Z_2 e^2}{x}\,\theta(-x)\,,$$

where $\theta(x)$ is the step function defined by

$$\theta(x) \equiv \begin{cases} 1 & \text{if } x > 0\,, \\ 0 & \text{if } x < 0\,. \end{cases}$$

Assuming that a particle is incident on the left with velocity v, what is the probability that it reaches the origin? Solve this problem by using the WKB approximation for the transmission coefficient obtained in part (b), assuming that it is possible to regard $x_2 = 0$ as the second turning point. Discuss the validity of the approximation. Is the use of the formula of part (b) strictly correct in this problem?

5. Shankar, problem 16.2.5 on p. 449. Consider the $\ell = 0$ radial equation for the three-dimensional Coulomb problem. Since $V(r)$ is singular at the turning point $r = 0$, we cannot use $n + \frac{3}{4}$ as in eq. (16.2.42) of Shankar.

(a) Will the additive constant be more or less than $\frac{3}{4}$? Explain your reasoning.

(b) By analyzing the exact equation near $r = 0$, it can be shown that the constant equals 1. Using this constant, show that the WKB energy levels agree with the exact results.