DUE: TUESDAY, APRIL 27, 2010

1. We wish to determine the correct form of the Schrodinger equation for a spin-$\frac{1}{2}$ particle in an external electromagnetic field. The wavefunction for a spin-$\frac{1}{2}$ particle is a two-component "spinor" wave function, where each component is an independent function of $\vec{\boldsymbol{x}}$ and t. Thus, the Hamiltonian must be a 2×2 matrix, whose elements are also operators on the Hilbert space of square integrable functions.

(a) First consider a free spin-$\frac{1}{2}$ particle. We demand that H is rotationally invariant. Two candidate Hamiltonians are:

$$(i)\quad H_1 = \frac{\vec{\boldsymbol{p}}^2}{2m}\mathbf{I}\,, \qquad\qquad (ii)\quad H_2 = \frac{(\vec{\boldsymbol{\sigma}}\boldsymbol{\cdot}\vec{\boldsymbol{p}})^2}{2m}\,.$$

where $\mathbf{I}$ is the 2×2 identity matrix. Show that H_1 and H_2 are in fact identical.

(b) Consider now a spin-$\frac{1}{2}$ particle with charge q in an external electromagnetic field. Using the principle of minimal coupling, deduce the Hamiltonians corresponding to (i) and (ii) above, which include the dependence on the scalar potential $\phi(\vec{\boldsymbol{x}}, t)$ and the vector potential $\vec{\boldsymbol{A}}(\vec{\boldsymbol{x}}, t)$ [do *not* choose a specific gauge]. Show that the two resulting Hamiltonians differ. In particular, in case (ii) above, a term in the Hamiltonian arises of the form

$$\frac{ge}{2mc}\vec{\boldsymbol{S}}\boldsymbol{\cdot}\vec{\boldsymbol{B}}\,,$$

where g is called the "g"–factor and $\vec{\boldsymbol{S}} \equiv \frac{1}{2}\hbar\vec{\boldsymbol{\sigma}}$ is the spin operator. Using the fact that the electron has $q = -e$, what value of g is predicted by this approach?

2. The current density $\vec{\boldsymbol{j}}$ (in the absence of an external electromagnetic field) was introduced by Shankar in eq. (5.3.8) on p. 166. The corresponding number density is given by $\rho = |\psi(\vec{\boldsymbol{r}})|^2$. Both quantities are related through the continuity equation [eq. (5.3.2) of Shankar on p. 165]. It is convenient to redefine these quantities by multiplying them by the charge e of the particle. Then, the corresponding number density is reinterpreted as the charge density and the corresponding current density is reinterpreted as the electromagnetic current density.

(a) Show that the electromagnetic current density for a particle with charge e in an external electromagnetic field is given by:

$$\vec{\boldsymbol{j}} = \frac{e\hbar}{2im}\left(\psi^*\vec{\boldsymbol{\nabla}}\psi - \psi\vec{\boldsymbol{\nabla}}\psi^*\right) - \frac{e^2}{mc}\vec{\boldsymbol{A}}\psi^*\psi = \frac{e}{m}\mathrm{Re}\left[\psi^*\left(\vec{\boldsymbol{p}} - \frac{e\vec{\boldsymbol{A}}}{c}\right)\psi\right]\,,$$

where $\vec{\boldsymbol{A}}$ is the electromagnetic vector potential and $\vec{\boldsymbol{p}} = -i\hbar\vec{\boldsymbol{\nabla}}$.

(b) Show that the continuity equation remains valid. Is the definition of ρ modified as compared to the case of zero external electromagnetic field?

3. Consider a spinless particle subject to an isotropic three dimensional harmonic oscillator potential. The Hamiltonian for this problem in the absence of an external electromagnetic field is given by:

$$H_0 = \frac{\vec{\boldsymbol{p}}^2}{2\mu} + \tfrac{1}{2}\mu\omega_0^2\vec{\boldsymbol{r}}^2\,, \qquad \text{where} \ \ \vec{\boldsymbol{r}}^2 \equiv x^2+y^2+z^2\,,$$

where ω_0 is a positive constant and μ is the mass of the particle with charge q. We now turn on a uniform magnetic field B parallel to the z axis. Define

$$\omega_L \equiv -\frac{qB}{2\mu c}\,.$$

(a) Write down the Hamiltonian in the Coulomb gauge in the form:

$$H = H_0 + H_1(\omega_L)\,,$$

where H_1 is the sum of an operator that depends linearly on ω_L (the paramagnetic term) and an operator which depends quadratically on ω_L (the diamagnetic term). First compute the energy eigenstates with $B=0$. In the presence of a uniform magnetic field B parallel to the z axis, show that the new eigenstates of the system and their degeneracies can be determined exactly. Compute the energy eigenvalues (and their corresponding degeneracies) explicitly for arbitrary ω_L.

(b) Show that if $\omega_L \ll \omega_0$, then the paramagnetic term dominates over the diamagnetic term.

(c) Consider the first excited state of the oscillator, i.e. the states whose energies approach $5\hbar\omega_0/2$ as $\omega_L \to 0$. Using the results of part (a) what are the energy levels in the presence of the B field and their degrees of degeneracy, to first order in ω_L/w_0? Sketch the energy levels as a function of B.

(d) Now consider the ground state. How does its energy vary as a function of ω_L? Is the ground state in the presence of the B–field and eigenvector of $\vec{\boldsymbol{L}}^2$? of L_z? of L_x? Give the form of the wavefunction and the corresponding probability current. Show that the effect of the B–field is to compress the wave function about the z–axis in a ratio $[1+(\omega_L/\omega_0)^2]^{1/4}$ and to induce a current.

4. Consider a charged particle (with charge q) whose motion is confined to a circle of radius R in the x–y plane, with its center at the origin. A thin magnetic flux tube of radius $r<R$ is located with its axis along the z-axis. The magnetic field is confined within the flux tube, and the total magnetic flux through the x–y plane is denoted by Φ. In particular, the charged particle moves in a region where there is no magnetic field. It is convenient to work in cylindrical coordinates (ρ,θ,z), where $x=\rho\cos\theta$ and $y=\rho\sin\theta$. In the region where there is no magnetic field, $\vec{\boldsymbol{\nabla}}\times\vec{\boldsymbol{A}}=0$, which implies that

$$\vec{\boldsymbol{A}}(\rho,\theta,z) = \vec{\boldsymbol{\nabla}}\chi(\rho,\theta,z)\,. \tag{1}$$

(a) Noting that Stokes' theorem relates Φ to the line integral of $\vec{\boldsymbol{A}}$ taken along the circle of radius R, show that the choice,

$$\chi(\rho, \theta, z) = \frac{\Phi\theta}{2\pi},$$

satisfies Stokes' theorem and the Coulomb gauge condition.

HINT: Insert the value of χ into eq. (1) and evaluate $\vec{\boldsymbol{A}}$. Show that the vector potential points in the $\hat{\boldsymbol{\theta}}$ direction.

(b) The wave function for the charged particle is only a function of θ (since $\rho = R$ and $z = 0$ are fixed due to the confined motion). Write down the time-independent Schrodinger equation for the charged particle wave function $\psi(\theta)$ in the cylindrical coordinate representation (simplifying your equation as much as possible).

(c) Solve the Schrodinger equation of part (b) for the energy eigenvalues and eigenfunctions. Show that the allowed energies depend on Φ even though the charged particle on the circle never encounters the magnetic field.

HINT: Show that the energy eigenstates are also eigenstates of $\partial/\partial\theta$.

5. The hydrogen atom is placed in a weak uniform electric field of strength $\mathcal{E}$ pointing in the z-direction. The Hamiltonian describing the system is given by:

$$H = \frac{-\hbar^2}{2m}\vec{\boldsymbol{\nabla}}^2 - \frac{e^2}{r} - e\mathcal{E}z\,.$$

Compute the ground state energy of the system using the variational technique. Use the trial wave function

$$\psi(\vec{\boldsymbol{r}}) = N(1 + q\mathcal{E}z)\psi_{100}(\vec{\boldsymbol{r}})\,,$$

where $\psi_{100}(\vec{\boldsymbol{r}})$ is the ground state wave function of the hydrogen atom (in the absence of an external electric field), q is the variational parameter, and N is chosen such that the trial wave function is properly normalized. Ignore all spin effects (*i.e.*, ignore fine and hyperfine splittings). Since the external electric field is assumed to be weak, simplify your computations by expanding in $\mathcal{E}$ and keeping only the leading term. In particular, show that the first correction to the ground state energy of hydrogen is proportional to $\mathcal{E}^2$.