DUE: TUESDAY, MAY 11, 2010

MIDTERM ALERT: There is a change of plans for the midterm exam. The midterm exam will be take place from 10–11:45 am on Thursday May 13, 2010 in ISB 231 (our usual classroom). The exam will cover material from the first five topics of the course syllabus and the first three problem sets. During the exam, you may consult Shankar and Baym, your class notes (and any other handwritten notes), and any of the homework solutions and class handouts that are posted on the course website.

1. A system of three unperturbed states consisting of a degenerate pair of states of energy E_1 and a non-degenerate state of energy E_2 is subsequently perturbed, and is represented by the Hamiltonian matrix:

$$\begin{pmatrix} E_1 & 0 & a \\ 0 & E_1 & b \\ a^* & b^* & E_2 \end{pmatrix} , \tag{1}$$

where $E_2 > E_1$. The quantities a and b are to be regarded as perturbations that are of the same order but small compared with $E_2 - E_1$.

(a) Use second order non–degenerate perturbation theory to calculate the perturbed eigenvalues. Is this procedure correct?

(b) Use second order degenerate perturbation theory to calculate the perturbed eigenvalues.

(c) Calculate the eigenvalues exactly and compare with the results of parts (a) and (b).

2. A diatomic molecule behaves like a rigid rotator with a moment of inertia $I = Mr_0^2$, where M is the reduced mass and r_0 is the distance between the atoms. The Hamiltonian can be approximated by:

$$H = \frac{\vec{\boldsymbol{L}}^2}{2I} . \tag{2}$$

Assume that the molecule consists of two atoms of charge $\pm e$, separated by a distance r_0. Suppose that a uniform electric field is present which points in a fixed direction. Compute the energy eigenvalues of the system, treating the electric field as a perturbation. Use second order perturbation theory.

HINT: There is one hard integral. This can be done trivially, if you know the following Clebsch-Gordon coefficient:

$$\langle \ell\; m\,,\, 1\; 0 \,|\, \ell - 1\; m\,,\, \ell\; 1\rangle = -\left[\frac{\ell^2 - m^2}{\ell(2\ell+1)}\right]^{1/2} .$$

3. Positronium is a bound state of two spin-1/2 particles: an electron (e^-) and a positron (e^+). Consider the Hamiltonian for the system, where we focus only on the spin degrees of freedom. In the presence of a uniform external magnetic field, we may take:

$$H = A(1 - \vec{\boldsymbol{\sigma}}_{\mathbf{1}}\boldsymbol{\cdot}\vec{\boldsymbol{\sigma}}_{\mathbf{2}}) + \mu_B(\vec{\boldsymbol{\sigma}}_{\mathbf{1}} - \vec{\boldsymbol{\sigma}}_{\mathbf{2}})\boldsymbol{\cdot}\vec{\boldsymbol{B}}\,,$$

where A is a constant and μ_B is the Bohr magneton. The labels 1 and 2 refer to the electron and positron respectively.

(a) In zero magnetic field, a transition is observed to occur from the ${}^3S_1$ state to the ${}^1S_0$ state (which is the ground state). The emitted photon is observed to have a frequency of 2×10^5 MHz. Evaluate the constant A.

(b) Now turn on the magnetic field. Assume it points in the z-direction. Treating the magnetic field as a perturbation, compute the energy eigenvalues to second order in B and the energy eigenstates to first order in B.

(c) Repeat the calculation of part (b), but now solve the problem exactly. Expand out your solutions in a power series in B, and verify that the results of part (b) are indeed correct.

4. Consider the hydrogen atom in the $n = 2$ state. It is placed in a uniform magnetic field B. The Hamiltonian for the hydrogen atom in the non-relativistic limit is a sum of the Coulomb interaction with minimal coupling, the fine structure and the hyperfine structure contributions. That is,

$$H = H_{\rm C} + H_{\rm FS} + H_{\rm HFS}$$

For this problem the hyperfine structure will be ignored. The fine structure Hamiltonian is given by:

$$H_{\rm FS} = m_e c^2 - \frac{\vec{\boldsymbol{p}}^{\,4}}{8m_e^3c^2} + \frac{1}{2m_e^2c^2 r}\frac{dV}{dr}\vec{\boldsymbol{L}}\boldsymbol{\cdot}\vec{\boldsymbol{S}} + \frac{\hbar^2}{8m_e^2c^2}\vec{\boldsymbol{\nabla}}^{\mathbf{2}}V(\vec{\boldsymbol{r}})\,.$$

Here $\vec{\boldsymbol{L}}$ and $\vec{\boldsymbol{S}}$ are the orbital and spin angular momentum of the electron.

(a) Using first order degenerate perturbation theory, compute the energy levels as a function of B, assuming that the contributions due to the fine structure and due to the term in the Hamiltonian which is linear in B are roughly of equal strength. You may neglect the term in the Hamiltonian which is quadratic in B. Also, neglect the proton's spin and the associated hyperfine structure.

HINT: You first need to employ the results of problem 1 of problem set #2 for the spin-dependent Hamiltonian of an electron in an external magnetic field. You will also need the first order splitting due to the fine structure Hamiltonian [cf. eq. (17.3.22) on p. 469 of Shankar],

$$E_{\rm FS}^{(1)} = -\frac{m_e c^2\alpha^2}{2n^2}\cdot\frac{\alpha^2}{n}\left(\frac{1}{j+1/2} - \frac{3}{4n}\right),$$

where j is the total angular momentum. Finally, the following Clebsch-Gordon coefficients may be useful:

$$\langle \ell\ m_\ell\,,\ \tfrac{1}{2}\ m_s\,|\,\ell+\tfrac{1}{2}\ m\,,\ \ell\ \tfrac{1}{2}\rangle = \left[\frac{\ell+2mm_s+\frac{1}{2}}{2\ell+1}\right]^{1/2},$$

$$\langle \ell\ m_\ell\,,\ \tfrac{1}{2}\ m_s\,|\,\ell-\tfrac{1}{2}\ m\,,\ \ell\ \tfrac{1}{2}\rangle = (-1)^{m_s+\frac{1}{2}}\left[\frac{\ell-2mm_s+\frac{1}{2}}{2\ell+1}\right]^{1/2},$$

where $m = m_\ell + m_s$.

(b) Compute the limits of large and small B.

(c) Sketch a picture of the energy levels found in part (a) as a function of B.

5. Consider the hydrogen atom in an excited $n = 2$ state, which is subjected to an external uniform electric field E. Do *not* neglect the spin of the electron. Assume that the field E is sufficiently weak so that eEa_0 is small compared to the fine structure, but such that the Lamb shift δ ($\delta = 1057$ MHz) cannot be neglected. That is, treat the problem as a two-level system consisting of the the $2S_{1/2}$ and $2P_{1/2}$ states of hydrogen In particular, you may ignore the $2P_{3/2}$ state of hydrogen and the hyperfine interactions.

(a) Compute the Stark effect for the $2S_{1/2}$ and $2P_{1/2}$ states of hydrogen by solving the two-level system exactly.

HINT: When electron spin is included, the hydrogen atom energy eigenstates are two-component wave functions given (in the coordinate representation) by:

$$\psi(\vec{\boldsymbol{r}}) = R_{n\ell}(r)\mathcal{Y}^j_{\ell m}(\theta,\phi)\,,$$

where $R_{n\ell}(r)$ is the radial wave function of the hydrogen atom, and the spin spherical harmonics are defined by:[1]

$$\mathcal{Y}^{j=\ell\pm\frac{1}{2}}_{\ell m}(\theta,\phi) \equiv \langle\theta,\phi|j=\ell\pm\tfrac{1}{2},m\rangle\,,$$

where

$$\left|j=\ell\pm\tfrac{1}{2},m\right\rangle = \frac{1}{\sqrt{2\ell+1}}\left[\pm(\ell+\tfrac{1}{2}\pm m)^{1/2}\left|\ell,\tfrac{1}{2};m-\tfrac{1}{2},\tfrac{1}{2}\right\rangle + (\ell+\tfrac{1}{2}\mp m)^{1/2}\left|\ell,\tfrac{1}{2};m+\tfrac{1}{2},-\tfrac{1}{2}\right\rangle\right].$$

(b) Show that for $eEa_0 \ll h\delta$, the energy shifts due to the external electric field are quadratic in E, whereas for $eEa_0 \gg h\delta$, they are linear in E. Determine the (perturbed) energy eigenstates in both limiting cases.

(c) The critical field is defined as:

$$E_c \equiv \frac{h\delta}{\sqrt{3}ea_0}\,,$$

where the factor of $\sqrt{3}$ is conventional. The linear or quadratic behavior of the energy shifts obtained in part (b) depend on the magnitude of E as compared to E_c. Determine E_c in volts/cm.

[1]Note that the usual spherical harmonics can be written as: $Y_{\ell m}(\theta,\phi) \equiv \langle\theta,\phi\,|\,\ell,m\rangle$.