DUE: MAY 25, 2010

1. (a) Consider the Born approximation as the first term of the Born series. Show that:

(i) the Born approximation for the *forward* scattering amplitude [i.e. at $\theta = 0$] is purely real, and therefore

(ii) the Born approximation fails to satisfy the optical theorem.

Do not assume that the potential is spherically symmetric. However, you may assume that the potential is hermitian.

(b) Consider the Yukawa potential:

$$V = -\frac{ge^{-\mu r}}{r}\,.$$

In class, we computed the (first) Born approximation to the scattering amplitude. Consider now the second Born approximation; i.e., the second term in the Born series. Compute the scattering amplitude *in the forward direction*, $\theta = 0$, in the second Born approximation.[1] Check to see whether the optical theorem is now satisfied.

HINT: You will need to evaluate $\langle\vec{\boldsymbol{k}}|V(E - H_0 + i\epsilon)^{-1}V|\vec{\boldsymbol{k}}\rangle$, where $H_0 = \vec{\boldsymbol{P}}^{\,2}/(2m)$. In class, we inserted a complete set of position eigenstates in order to convert this matrix element as a multiple integral over $d^3\boldsymbol{r}_1 d^3\boldsymbol{r}_2$. However, it is easier to evaluate the matrix element by inserting a complete set of momentum eigenstates, $|\boldsymbol{k}'\rangle$. You will then only have to evaluate an integral over $d^3\boldsymbol{k}'$.

(c) Compare the magnitudes of the first and second terms of the Born series for the forward scattering amplitude. What condition do you find if you require the second term in the Born series to be smaller than the first term? Compare this condition with the one you would get for the validity of the Born approximation based on the formula derived in class.

(d) Using the first Born approximation for the scattering amplitude, compute the s and p wave phase shifts. Under what circumstances does the s-wave phase shift dominate? Is the Born approximation valid in this case?

2. Consider the scattering of particles by the square well potential in three dimensions:

$$V(r) = \begin{cases} -V_0\,, & \text{for } r < a\,, \\ 0\,, & \text{for } r > a\,, \end{cases} \tag{1}$$

[1] Do not attempt to compute the scattering amplitude in the second Born approximation for $\theta \neq 0$. It is extremely messy!

where V_0 is positive.

(a) Obtain the differential cross-section in the Born approximation.

(b) Using the results of part (a), evaluate the total cross-section in the limits of low and high energy. Specifically, show that at low energies, the cross section can be approximated by:

$$\sigma \simeq \sigma_0(1 + Ak^2)\,,$$

where the energy $E = \hbar^2k^2/(2m)$. You should evaluate the constants σ_0 and A. In the high energy limit, show that

$$\sigma \simeq \frac{C}{k^2}\,,$$

where the constant C should be determined.

HINT: To determine the total cross-section in the high-energy limit, you should convert the integral to a manageable form before making any approximations. First integrate over the azimuthal angle. Then, change variables from $\cos\theta$ to $y = 2ka\sin(\theta/2) = ka[2(1-\cos\theta)]^{1/2}$, and express the total cross-section as an integral over y. Now, you can evaluate the integral by taking the infinite energy limit. However, the resulting integral is difficult (warning—Mathematica will have a very hard time with this, although Maple can do it quite easily!). So, I will help you out by providing the following result:

$$\int_0^\infty \frac{[j_n(y)]^2}{y^p}\,dy = \frac{2^{p-2}\,\Gamma\left(\dfrac{2n+1-p}{2}\right)\Gamma^2\left(\dfrac{p+1}{2}\right)}{\Gamma(p+1)\,\Gamma\left(\dfrac{2n+p+3}{2}\right)}\,, \qquad (-1 < \operatorname{Re} p < 2n+1)\,.$$

where $j_n(y)$ is a spherical Bessel function and $\Gamma^2(z)$ is the square of the gamma function. Show that the integral you are trying to evaluate corresponds to a specific choice of n and p above. Then evaluate it.

(c) What is the range of validity of your answers to parts (a) and (b). Consider separately the limits of low and high energy.

(d) Using the results of part (a), it is possible to perform the integral exactly and obtain an expression for the total cross-section. Obtain the exact formula for the cross-section in the Born approximation. Then, check the low and high energy limits and confirm the results of part (b).

(e) Compute the s-wave phase shift for the scattering of particles by the attractive square well potential given in eq. (1). At low energies, $E = \hbar^2k^2/(2m)$, one may neglect the higher partial waves. Hence, for $ka \ll 1$, one can assume that only the s-wave scattering is important. In this limit, compute the differential and total cross-section for the scattering of particles by the square well potential as a function of k_1, where $E + V_0 = \hbar^2k_1^2/(2m)$. Compare these results to the results of parts (a) and (b). In what limit do the results for the cross-sections coincide?

3. Consider the case of low-energy scattering from a spherical delta-function shell,

$$V(r) = V_0 \delta(r - a)\,,$$

where V_0 and a are constants. Calculate the scattering amplitude, $f(\theta)$, the differential cross-section and the total cross-section, under the assumption that $ka \ll 1$, so that only s-wave scattering is important.

HINT: Solve the time-independent Schrodinger equation exactly in the case of $\ell = 0$ for the radial wave function, $R(r) \equiv u(r)/r$. Consider separately the cases of $r < a$ and $r > a$. By integrating the Schrodinger equation from $r = a - \epsilon$ to $a + \epsilon$ (where $0 < \epsilon \ll 1$), show that

$$\left[\left.\frac{du}{dr}\right|_{a+\epsilon} - \left.\frac{du}{dr}\right|_{a-\epsilon} \right] = \frac{2mV_0}{\hbar^2} u(a)\,.$$

Inserting your explicit solutions for $u(r)$ for the two cases $r < a$ and $r > a$ into the equation above, you should be able to determine the s-wave phase shift. In particular, find an expression for $\tan\delta_0$ in terms of V_0 and the wave number k. Evaluate the phase shift in the limit of $ka \ll 1$ to simplify your expression and then complete the problem.

4. Low energy scattering is parameterized by two parameters: the *scattering length* a and the *effective range* r_0. In this problem, you will verify this statement.

(a) Show that in the limit of $k \to 0$, (more precisely, for $kb \ll 1$, where b is the range of the potential $V(r)$, i.e. $V(r) \approx 0$ for $r > b$):

$$k \cot\delta_0(k) = -1/a\,,$$

where a is a parameter with units of length and $\delta_0(k)$ is the s-wave phase shift. What is the cross-section in the limit of zero energy?

(b) Obtain an expression for the partial wave amplitude:

$$a_0(k) = \frac{e^{2i\delta_0} - 1}{2ik}$$

in the limit of $k \to 0$, using the results of part (a). For what values of k does $a_0(k)$ have poles? Can one associate these poles with the existence of bound states? What is the relation between the bound state energy E_b, and the scattering length? Obtain an expression for the total cross-section as a function of the energy $E = \hbar^2 k^2/(2m)$, assuming that the low energy approximation is still valid. Express your result in terms of E_b.

(c) Show that by considering the radial integral equation:

$$\text{(i)} \quad G^{\ell}_{-k}(r,r') = G^{\ell}_{k}(r,r')^*\,,$$

$$\text{(ii)} \quad A_\ell(-k,r) = (-1)^\ell A_\ell(k,r)^*\,,$$

$$\text{(iii)} \quad \exp(2i\delta_\ell(k)) = \exp(-2i\delta_\ell(-k))\,.$$

You will need (i) and (ii) to prove (iii). Using (iii), show that $\cot\delta_\ell(k)$ is an odd function of k. Assuming it has a power series about $k=0$, show that:

$$k^{2\ell+1}\cot\delta_\ell(k) = \frac{-1}{a_\ell} + \tfrac{1}{2}r_\ell k^2 + \mathcal{O}(k^4)\ .$$

For the case of $\ell = 0$, show that a_0 and r_0 each have dimensions of length.

(d) Obtain expressions for the partial wave amplitude $a_0(k)$ and low energy cross-section in terms of the scattering length $a \equiv a_0$ and the effective range r_0 which appear in the expansion obtained in part (c).

5. In this problem, I will lead you through the steps involved in solving the scattering problem for a charged particle subject to the Coulomb potential. We shall first solve the Schrodinger equation,

$$\left(-\frac{\hbar^2\vec{\boldsymbol{\nabla}}^2}{2m} - \frac{Ze^2}{r}\right)\psi(\vec{\boldsymbol{r}}) = E\psi(\vec{\boldsymbol{r}})\,, \qquad \text{for}\quad E>0\,. \tag{2}$$

(a) Define the dimensionless quantity,

$$\gamma \equiv -\frac{mZe^2}{\hbar^2 k}\,.$$

Let $\psi(\vec{\boldsymbol{r}}) = e^{i\vec{\boldsymbol{k}}\cdot\vec{\boldsymbol{r}}}X(\vec{\boldsymbol{r}})$, with $E \equiv \hbar^2k^2/(2m)$. Inserting this result into eq. (2), derive the following differential equation for X,

$$\vec{\boldsymbol{\nabla}}^2 X + (2i\vec{\boldsymbol{k}}\cdot\vec{\boldsymbol{\nabla}})X - \frac{2\gamma k}{r}X = 0\,. \tag{3}$$

(b) Set up the coordinate system so that the beam is incoming along the z-direction. Define a new variable,

$$u = kr - \vec{\boldsymbol{k}}\cdot\vec{\boldsymbol{r}} = kr(1-\cos\theta)\,.$$

Show that eq. (3) now becomes

$$u\frac{d^2X}{du^2} + (1-iu)\frac{dX}{du} - \gamma X = 0\,. \tag{4}$$

Solve this equation, subject to the boundary condition that the solution for $\psi(\vec{\boldsymbol{r}})$ must be non-singular at the origin. Feel free to consult your favorite book on special functions of mathematical physics.[2] Show that the solution to eq. (4) is a confluent hypergeometric function,

$$X(u) = C\,{}_1F_1(-i\gamma, 1; iu)\,,$$

where C is a constant to be determined.

(c) To determine the constant C, consider the asymptotic behavior of $\psi(\vec{\boldsymbol{r}})$ as $r \to \infty$. Show that one can choose C such that:

$$\psi(\vec{\boldsymbol{r}}) = \psi_{inc}(\vec{\boldsymbol{r}}) + \psi_{sc}(\vec{\boldsymbol{r}})\,,$$

where the incident wave function is

$$\psi_{inc}(\vec{\boldsymbol{r}}) = \exp\left\{ikz + i\gamma \ln[k(r-z)]\right\}\left(1 + \frac{\gamma^2}{ik(r-z)}\right)\,,$$

with $z = r\cos\theta$, and the scattered wave function is

$$\psi_{sc}(\vec{\boldsymbol{r}}) = \frac{\exp\left\{ikr - i\gamma \ln[k(r-z)]\right\}}{ik(r-z)}\,\frac{\Gamma(1+i\gamma)}{\Gamma(-i\gamma)}\,.$$

<u>*HINT:*</u> You will need to look up the asymptotic expansion for the confluent hypergeometric function ${}_1F_1(a, b; x)$ in the appropriate reference book (see footnote 1).

(d) Define the Coulomb scattering amplitude by:

$$\psi_{sc}(\vec{\boldsymbol{r}}) = \frac{e^{i[kr - \gamma\ln(2kr)]}}{r} f_c(\theta), \qquad \text{as} \;\; r \to \infty\,.$$

Obtain an explicit expression for $f_c(\theta)$. Express your answer in terms of the pure phase factor, $e^{2i\delta_0} \equiv \Gamma(1+i\gamma)/\Gamma(1-i\gamma)$.

(e) Compute the probability currents j_{inc} and j_{sc} and following the same procedure used in class, show that:

$$\frac{d\sigma}{d\Omega} = |f_c(\theta)|^2\,.$$

Using the expression for $f_c(\theta)$ obtained in part (d), compute the differential cross section and verify that your result coincides with the *Rutherford scattering formula.* Show that the total cross section σ diverges.

(f) Show that the poles of $f_c(\theta)$ correspond to the bound states of a hydrogenic atom with atomic number Z.

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[2] One of my favorites is N.N. Lebedev, *Special Functions and their Applications* (Dover Publications, Inc., New York, NY, 1972). The Dover books are generally not very expensive, and this book in particular is well worth the investment. Of course, you can solve eq. (4) using the standard series technique for solving differential equations, but this will require an additional investment in time.

(g) Expand $f_c(\theta)$ in partial waves:

$$f_c(\theta) = \sum_{\ell=0}^{\infty} (2\ell+1) P_\ell(\cos\theta) f_\ell \,. \tag{5}$$

Project out f_ℓ by expressing it as an integral[3] over $\cos\theta$. Note that this integral diverges for real values of γ. To circumvent this unfortunate fact, evaluate the integral for $\mathrm{Re}(-i\gamma) > 0$ and analytically continue the result to pure imaginary γ. Simplify the resulting expression using the appropriate gamma function identities. Verify that the following result is obtained:

$$f_\ell = \frac{1}{2ik}\, e^{2i\delta_\ell} \,,$$

where[4]

$$e^{2i\delta_\ell} \equiv \frac{\Gamma(\ell+1+i\gamma)}{\Gamma(\ell+1-i\gamma)} \,. \tag{6}$$

Note that $f_c(\theta)$ diverges at $\theta = 0$ (check this by showing that the series given by eq. (5) diverges when $\cos\theta = 1$). This is not surprising since σ diverges, which implies that the imaginary part of the forward scattering amplitude must also diverge due to the optical theorem.

(h) [*EXTRA CREDIT*] Finally, show that the Coulomb wave function $\psi(\vec{\boldsymbol{r}})$ can be expanded in partial wave and takes the form

$$\psi(\vec{\boldsymbol{r}}) = \frac{1}{kr} \sum_{\ell=0}^{\infty} (2\ell+1) i^\ell P_\ell(\cos\theta) e^{i\delta_\ell} F_\ell(\gamma, kr) \,,$$

where $F_\ell(\gamma, kr)$ exhibits the asymptotic behavior,

$$F_\ell(\gamma, kr) \longrightarrow \sin\bigl(kr - \gamma\ln(2kr) - \tfrac{1}{2}\ell\pi + \delta_\ell\bigr) \,, \qquad \text{as} \;\; r \to \infty \,,$$

which justifies the identification of δ_ℓ as the phase shift for the Coulomb scattering problem. Notice the appearance of the logarithm in the argument of the sine function. This is a remnant of the long-range nature of the Coulomb potential, which cannot be neglected even as $r \to \infty$.

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[3] After setting $z = \cos\theta$, the following integral, taken from I.S. Gradshteyn and I.M. Ryzhik, *Table of Integrals, Series, and Products*, 7th Edition (Elsevier Academic Press, Amsterdam, 2007), should be useful:

$$\int_{-1}^{1} (1-z)^p P_\ell(z)\,dz = \frac{(-1)^p\, 2^{p+1}\, [\Gamma(p+1)]^2}{\Gamma(p+\ell+2)\Gamma(1+p-\ell)} \,, \qquad \text{where} \;\; \mathrm{Re}\, p > -1 \,.$$

[4] In particular, note that for complex z, the gamma function satisfies the relation $\Gamma(z^*) = \Gamma(z)^*$. Hence, eq. (6) is the ratio of a complex number and its complex conjugate, which must be a pure phase.