DUE: JUNE 8, 2010

FINAL EXAM ALERT: The final exam will be take place from 11 am–2 pm on Wednesday June 9, 2010 in ISB 235. Please note the change of date, time and location. The exam will cover the entire course. During the exam, you may consult Shankar and Baym, your class notes (and any other handwritten notes), and any of the homework solutions and class handouts that are posted on the course website.

1. Tritium (the isotope H^3), which is initially in its ground state, undergoes spontaneous beta decay, emitting an electron of maximum energy of about 17 keV. The nucleus remaining is He^3.

(a) Calculate the probability that the electron of this ion is left in a quantum state of principal quantum number $n = 2$.

(b) What is the probability that the electron of this ion is left in quantum state with $\ell \neq 0$?

In this problem, you should neglect nuclear recoil. Note the energy of the emitted electron. What is the relevant approximation? Explain.

2. Consider a two-level system with $E_1 < E_2$. There is a time-dependent potential that connects the two levels as follows:

$$V_{11} = V_{22} = 0, \qquad V_{12} = \gamma e^{i\omega t}, \qquad V_{21} = \gamma e^{-i\omega t} \qquad (\gamma \ \text{ real}),$$

where $V_{ij} = \langle i|V|j\rangle$. At time $t = 0$, it is known that only the lower level is populated—that is, $c_1(0) = 1$ and $c_2(0) = 0$. Note that a general state of the system can be expressed as a linear combination of eigenstates of the unperturbed Hamiltonian (in the Schrodinger picture):

$$|\psi(t)\rangle = \sum_{n=1}^{2} c_n(t) e^{-iE_n t/\hbar} |n\rangle$$

(a) Starting with the time-dependent Schrodinger equation, derive the following differential equation for $c_k(t)$:

$$i\hbar \frac{dc_k}{dt} = \sum_{n=1}^{2} V_{kn}(t) e^{i\omega_{kn} t} c_n \,, \qquad (k = 1, 2)\,, \tag{1}$$

where $V_{kn}(t) \equiv \langle k|V(t)|n\rangle$ and $\hbar\omega_{kn} \equiv E_k - E_n$. By solving the above system of differential equations *exactly*, find $|c_1(t)|^2$ and $|c_2(t)|^2$ for $t > 0$.

HINT: It is convenient to define new coefficients,

$$c_1'(t) \equiv e^{i(\omega_{21}-\omega)t/2}\, c_1(t)\,, \qquad\qquad c_2'(t) \equiv e^{-i(\omega_{21}-\omega)t/2}\, c_2(t)\,.$$

Then, show that eq. (1) reduces to a matrix differential equation of the form

$$i\hbar\frac{d}{dt}\begin{pmatrix} c_1'(t) \\ c_2'(t)\end{pmatrix} = A\begin{pmatrix} c_1'(t) \\ c_2'(t)\end{pmatrix}\,, \tag{2}$$

where A is a *time-independent* 2×2 traceless hermitian matrix. Verify that the solution to eq. (2) is

$$\begin{pmatrix} c_1'(t) \\ c_2'(t)\end{pmatrix} = e^{-iAt/\hbar}\begin{pmatrix} c_1'(0) \\ c_2'(0)\end{pmatrix}\,.$$

By writing $A = \vec{\boldsymbol{a}}\boldsymbol{\cdot}\vec{\boldsymbol{\sigma}}$ (where the vector $\vec{\boldsymbol{a}}$ is uniquely determined), it is straightforward to compute $e^{-iAt/\hbar}$ and complete part (a) of the problem.

(b) Do the same problem using time-dependent perturbation theory to lowest nonvanishing order. Compare the two approaches for small values of γ. Treat the following two cases separately: (i) ω very different from ω_{21}, and (ii) ω close to ω_{21}.

3. This problem provides a crude model for the photoelectric effect. Consider the hydrogen atom in its ground state (you may neglect the spins of the electron and proton). At time $t=0$, the atom is placed in a high frequency uniform electric field that points in the z-direction,

$$\vec{\boldsymbol{\mathcal{E}}}(t) = \mathcal{E}_0\hat{\boldsymbol{z}}\sin\omega t\,.$$

We wish to compute the transition probability per unit time that an electron is ejected into a solid angle lying between Ω and $\Omega + d\Omega$.

(a) Determine the minimum frequency, ω_0, of the field necessary to ionize the atom.

(b) Using Fermi's golden rule for the transition rate at first-order in time-dependent perturbation theory, obtain an expression for the transition rate per unit solid angle as a function of the polar angle θ of the ejected electron (measured with respect to the direction of the electric field).

HINT: The matrix element that appears in Fermi's golden rule describes a transition of the negative-energy bound electron in its ground state to a positive-energy "free" electron. The wave function of the latter is actually quite complicated, since one cannot really neglect the effects of the long-range Coulomb potential. Nevertheless, you should simplify the computation by assuming the wave function of the ejected electron is a free-particle plane wave, with wave number vector $\vec{\boldsymbol{k}}$. (Note that the direction of $\vec{\boldsymbol{k}}$ corresponds to that of the ejected electron).

(c) Integrate the result of part (b) over all solid angles to obtain the total ionization rate as a function of the frequency of the field. Determine the value of ω [in terms of ω_0 obtained in part (a)] for which the total ionization rate is maximal.

4. Consider the spontaneous emission of an $E1$ photon by an excited atom. The magnetic quantum numbers (m and m') of the initial and final atomic state are measured with respect to a fixed z-axis. Suppose the magnetic quantum number of the atom decreases by one unit.

(a) Compute the angular distribution of the emitted photon.

(b) Determine the polarization of the photon emitted in the z-direction.

(c) Verify that the result of part (b) is consistent with angular momentum conservation for the whole (atom plus photon) system.

HINT: The material on pp. 282–285 of Baym should be helpful.

5. Consider the elastic scattering of photons off electrons in atoms, assuming that the incident photon energies are large compared to the atomic binding energies. However, you should assume that the photon wavelength is still substantially larger than a typical atomic radius.

(a) Using the quantum theory of radiation, argue that the $\vec{\boldsymbol{A}}$ field operator must occur at least twice in the matrix element in order that there be a non-zero contribution in perturbation theory.

(b) Recall that there is a quadratic $\vec{\boldsymbol{A}}\boldsymbol{\cdot}\vec{\boldsymbol{A}}$ term in the interaction Hamiltonian.[1] Compute the differential cross-section to first order perturbation theory in the dipole approximation. Show that:

$$\frac{d\sigma}{d\Omega} = r_0^2 |\vec{\boldsymbol{\epsilon}}_\lambda \boldsymbol{\cdot} \vec{\boldsymbol{\epsilon}}^{\,*}_{\lambda'}|^2 \,,$$

where $r_0 \equiv e^2/(mc^2)$ is the classical radius of the electron.

(c) Compute the total cross-section, assuming that the initial photon beam is unpolarized and the polarization of the final state photon is not measured.

[1] One can show that this term, taken at first order in the perturbation expansion, will dominate the second order contribution due to the $\vec{\boldsymbol{j}}\boldsymbol{\cdot}\vec{\boldsymbol{A}}$ term of the interaction Hamiltonian, assuming as above that the energy of the incident photon is large. (*EXTRA CREDIT:* Verify this statement.)