FINAL EXAM INSTRUCTIONS: This is an open book exam. You are permitted to consult the textbooks of Shankar and Baym, your handwritten notes, and any class handouts that are posted to the course website. One mathematical reference book is also permitted. No other consultations or collaborations are permitted during the exam. *In order to earn total credit for a problem solution, you must show all work involved in obtaining the solution.* However, you are *not* required to re-derive any formulae that you cite from the textbook or the class handouts. The point value of each problem is indicated in the square brackets below.

1. [30] An electron is placed in a potential

$$V(\vec{\boldsymbol{r}}) = \frac{-e^2}{r} + \beta(r^2 - 3z^2)\,,$$

where β is a small parameter. Neglect the spin of the electron.

(a) Compute the shifts of the $n = 2$ energy levels (you may neglect fine-structure effects) using first order perturbation theory. Indicate the relative positions of the energy levels.

(b) Suppose that a weak uniform magnetic field B is applied in the z-direction. Determine its effect on the levels obtained in part (a) to first order in B.

(c) Repeat part (b) assuming that the weak uniform magnetic field is applied in the x-direction.

<u>*HINTS*</u>: When evaluating matrix elements, think before you calculate. Often, you can show that a particular matrix element is zero without fully computing it. In part (c), use first order degenerate perturbation theory by taking the "unperturbed" energy eigenstates and eigenvalues to be the shifted $n = 2$ levels obtained in part (a).

2. [20] Consider a positively-charged spin-1/2 particle in an external magnetic field, governed by the Hamiltonian:

$$H = H_0\,\mathbf{I} - \gamma\vec{\boldsymbol{B}}\boldsymbol{\cdot}\vec{\boldsymbol{S}}\,,$$

where $\mathbf{I}$ is the identity operator in spin space, $\vec{\boldsymbol{S}}$ is the vector of spin-1/2 spin matrices, and γ is a constant (for a positively-charged particle, $\gamma > 0$). H_0 is spin-independent and is independent of the magnetic field $\vec{\boldsymbol{B}}$. For simplicity, assume that H_0 possesses exactly one eigenvalue, which is denoted by E.

(a) If the magnetic field is given by $\vec{\boldsymbol{B}} = B\hat{\boldsymbol{z}}$ (where $B > 0$), determine the energy eigenstates and eigenvalues of H.

(b) Assume that the magnetic field is given by $\vec{\boldsymbol{B}} = B\hat{\boldsymbol{z}}$ for time $t < 0$. The system is initially observed to be in a spin-up state. At $t = 0$, a time-dependent perturbation is added by modifying the magnetic field. The new magnetic field for $t > 0$ is given by:

$$\vec{\boldsymbol{B}} = b\,(\hat{\boldsymbol{x}}\cos\omega t - \hat{\boldsymbol{y}}\sin\omega t) + B\hat{\boldsymbol{z}}\,,$$

where $b > 0$. Using first-order time-dependent perturbation theory, derive an expression for the probability that the system will be found in a spin-down state at some later time $t = T$. For what range of values of ω is this result reliable?

3. [20] A spinless particle of charge e and mass m is bound to a three-dimensional harmonic oscillator potential of angular frequency ω_0. The particle is initially in the state $|n_x, n_y, n_z\rangle = |1, 1, 0\rangle$.

(a) Compute the lifetime of the state to decay by spontaneous emission of a photon, in the electric dipole approximation. Take $\hbar\omega_0 = 1$ Rydberg and $m = m_e$ (the mass of the electron), and evaluate the lifetime in seconds. Assume that the polarization of the emitted photon is not measured.

(b) Define θ and ϕ to be the polar and azimuthal angles of wave vector $\vec{\boldsymbol{k}}$ of the emitted photon with respect to the fixed z-axis. The wave function of the oscillator after the emission of the photon is of the form:

$$|\psi\rangle = N\sum_\lambda \bigl\{(\epsilon_\lambda^{\,*})_y\,|1,0,0\rangle + (\epsilon_\lambda^{\,*})_x\,|0,1,0\rangle\bigr\}\,, \tag{1}$$

where $\vec{\boldsymbol{\epsilon}}_\lambda = \bigl((\epsilon_\lambda)_x\,,\,(\epsilon_\lambda)_y\,,\,(\epsilon_\lambda)_z\bigr)$ is the polarization vector of the emitted photon and N is a normalization factor. Determine the normalization factor N as a function of θ by setting $\langle\psi|\psi\rangle = 1$. Then, compute the expectation value $\langle\psi|xy|\psi\rangle$ in terms of the angles θ, ϕ, and the fundamental constants of the problem.

4. [30] Consider the scattering of spinless particles in an attractive exponential spherically symmetric potential:

$$V(r) = -V_0 \exp(-r/r_0)\,,$$

with $V_0 > 0$. It is convenient to define two dimensionless variables for this problem: $\xi \equiv kr_0$ and $\eta \equiv 2mV_0 r_0^2/\hbar^2$, where $\hbar^2 k^2/(2m)$ is the energy of the incoming beam.

(a) Compute, the scattering amplitude and the differential and total cross sections, in the Born approximation, in terms of the variables ξ, η and r_0. Evaluate the total cross section in both the low and high energy limits.

(b) Determine the validity of the Born approximation at fixed energy. (That is, what relation must ξ and η satisfy?) What is the requirement in order that the Born approximation be valid at *all* energies?

(c) Using the scattering amplitude obtained in part (a), calculate the s-wave and p-wave phase shifts. [*NOTE*: it is sufficient to evaluate $e^{i\delta_\ell}\sin\delta_\ell$ for $\ell = 0$, 1.]

<u>*HINT*</u>: Expand the Born approximated scattering amplitude in a partial wave expansion, and use the orthogonality of the Legendre polynomials to obtain expressions for $e^{i\delta_\ell}\sin\delta_\ell$ for $\ell = 0$ and $\ell = 1$ in terms of ξ and η.

(d) Using the results of part (c), compute both the s-wave and p-wave phase shifts in the low and high energy limits. Sketch a graph of δ_ℓ/η as a function of ξ for $\ell = 0$ and $\ell = 1$. Do you find the expected behavior at low energies?

(e) Using the s-wave phase shift, compute the total cross section in the low energy limit, and show that the result is the same as obtained in part (a).