This sheet contains additional information to help you solve the problems of the final exam more efficiently.

1. For $\beta = 0$, the unperturbed $n = 2$ energy eigenfunctions are given by:

$$\psi_{200}(\vec{\boldsymbol{r}}) = \frac{1}{\sqrt{\pi}}\left(\frac{1}{2a_0}\right)^{3/2}\left(1 - \frac{r}{2a_0}\right) e^{-r/(2a_0)}\,,$$

$$\psi_{210}(\vec{\boldsymbol{r}}) = \frac{1}{2\sqrt{\pi}}\left(\frac{1}{2a_0}\right)^{3/2} \frac{r}{a_0}\, e^{-r/(2a_0)}\cos\theta\,,$$

$$\psi_{21\pm1}(\vec{\boldsymbol{r}}) = \mp\frac{1}{8\sqrt{\pi}}\left(\frac{1}{a_0}\right)^{3/2} \frac{r}{a_0}\, e^{-r/(2a_0)}\sin\theta\, e^{\mp i\phi}\,,$$

where $a_0 = \hbar^2/(me^2)$.

You will also need to compute some integrals. One useful integral is:

$$\int_0^\infty r^n\, e^{-r/a}\, dr = a^{n+1}\, n!\,,$$

where n is a non-negative integer.

Finally, the following matrix elements of L_x will be needed for part (c):

$$\langle \ell'\, m'_\ell|\, L_x\, |\ell\, m_\ell\rangle = \tfrac{1}{2}\hbar\delta_{\ell\ell'}\big\{[(\ell-m_\ell)(\ell+m_\ell+1)]^{1/2}\delta_{m'_\ell\,,\,m_\ell+1}+[(\ell+m_\ell)(\ell-m_\ell+1)]^{1/2}\delta_{m'_\ell\,,\,m_\ell-1}\big\}\,.$$

3. For the three-dimensional harmonic oscillator, the following matrix elements will be useful:

$$\langle n'_x|x|n_x\rangle = \left(\frac{\hbar}{2m\omega_0}\right)^{1/2}\left[\sqrt{n_x}\,\delta_{n'_x,n_x-1} + \sqrt{n_x+1}\,\delta_{n'_x,n_x+1}\right]\,,$$

and two similar expressions where x is replaced by y and z, respectively.

In your calculation of the lifetime, the following numerical data should be useful: 1 Ry $= 13.6$ eV, $m_e c^2 = 511$ keV, $\alpha \equiv e^2/(\hbar c) \simeq 1/137$, and $\hbar = 6.582\times10^{-16}$ eV$\cdot$sec.

In part (b), eq. (10) can be derived from time-dependent perturbation theory (using the electric dipole approximation). However, you can simply assume this form for the wave function without proof when working out the problem.

4. In this problem, you will need to perform a number of integrals. Please refer to the following brief table of indefinite integrals.

$$\int \frac{dx}{(a+bx)^2} = \frac{-1}{b(a+bx)} \, ,$$

$$\int \frac{x\, dx}{(a+bx)^2} = \frac{1}{b^2} \left[\ln|a+bx| + \frac{a}{a+bx} \right] \, .$$

$$\int \frac{dx}{(a+bx)^4} = \frac{-1}{3b(a+bx)^3} \, ,$$

In addition, a brief table of definite integrals is also provided below.

$$\int_0^\infty e^{-ar}\, r\, dr = \frac{1}{a^2} \, ,$$

$$\int_0^\infty e^{-ar} \sin(kr)\, dr = \frac{k}{a^2+k^2} \, ,$$

$$\int_0^\infty e^{-ar} \sin(kr)\, r\, dr = \frac{2ak}{(a^2+k^2)^2} \, .$$

Finally, here are a couple of sums useful for the low-energy expansions:

$$\frac{1}{1-z} = \sum_{n=0}^\infty z^n \, ,$$

$$\ln(1+z) = \sum_{n=1}^\infty \frac{(-1)^{n-1}\, z^n}{n} \, .$$