MIDTERM EXAM INSTRUCTIONS: This is an open book exam. You are permitted to consult the textbooks of Shankar and Baym, your handwritten notes, and any class handouts that are posted to the course website. One mathematical reference book is also permitted. No other consultations or collaborations are permitted during the exam. *In order to earn total credit for a problem solution, you must show all work involved in obtaining the solution.* However, you are *not* required to re-derive any formulae that you cite from the textbook or the class handouts. The point value of each problem is indicated in the square brackets below.

1. [40] Suppose we define

$$G(t) \equiv \int_{-\infty}^{\infty} dx K(x,t,;x,0) \tag{1}$$

where $K(x,t;x',t')$ is the propagator. Assume that the system has a time-independent Hamiltonian and a discrete energy level spectrum.

(a) Prove that the Fourier transform of G,

$$\widetilde{G}(E) \equiv \lim_{\epsilon\to 0} \frac{i}{\hbar}\int_0^{\infty} G(t)e^{iEt/\hbar}\, e^{-\epsilon t}\, dt \tag{2}$$

has poles at all the discrete energy levels of the system. Take ϵ to be a *positive* infinitesimal quantity.

HINT: For a time-independent Hamiltonian, the time-evolution operator has a simple form. Writing $K(x,t;x',t')$ as a coordinate-space matrix element of the time evolution operator, insert a complete set of energy eigenstates. Then, compute $G(t)$ and $\widetilde{G}(E)$.

(b) Consider the harmonic oscillator with a charge e in one dimension. You are asked to calculate the discrete energy levels of the system, when it is placed in a uniform electric field of strength $\mathcal{E}$. Compute the levels in three independent ways:

(i) using the WKB approximation;

(ii) using the path integral technique;

(iii) solving the Schrodinger equation exactly.

Do you expect the same result from all three methods?

HINTS: (i) In the WKB approximation, the following indefinite integral is useful:

$$\int \sqrt{ax^2+bx+c}\,dx = \frac{2ax+b}{4a}\sqrt{ax^2+bx+c} \;-\; \left(\frac{4ac-b^2}{8a}\right)\frac{1}{(-a)^{1/2}}\sin^{-1}\left(\frac{2ax+b}{\sqrt{b^2-4ac}}\right),$$

where $a<0$ and $b^2-4ac>0$.

(ii) In the path integral technique, proceed as follows. First, write the expression for the path integral. Next, by a change of variables, eliminate the linear term. The resulting expression is a phase factor times the path integral for the ordinary harmonic oscillator. Then, employ the results of part (a) above.

2. [40] In problem 4 of problem set #3, we considered the hydrogen atom in a uniform magnetic field which points in the $\hat{\boldsymbol{z}}$-direction. The energy levels were obtained as a function of B. This is the well-known Zeeman effect. However, the terms in the Hamiltonian that are quadratic in B were neglected. We now want to see the effect of including the latter. To make the analysis simple, you may ignore the effects of electron and nuclear spin (i.e., the fine-structure and hyperfine structure can be neglected).

(a) For simplicity, we shall first consider the $n = 1$ ground state of hydrogen. Evaluate the first-order energy shift due to a uniform magnetic field $\vec{\boldsymbol{B}} = B\hat{\boldsymbol{z}}$, assuming that the term in the Hamiltonian that is quadratic in B can be neglected.

(b) Compute the *quadratic* Zeeman effect for the ground-state hydrogen atom, due to the usually neglected $e^2\vec{\boldsymbol{A}}^{\mathbf{2}}/(2mc^2)$ term in the Hamiltonian taken to first order in perturbation theory. Assume that the external magnetic field is uniform and points in the $\hat{\boldsymbol{z}}$-direction. Writing the energy shift as $\Delta E \equiv -\frac{1}{2}\chi\vec{\boldsymbol{B}}^{\mathbf{2}}$, obtain an expression for the diamagnetic susceptibility, χ.

(c) How large a magnetic field is required in order that the two contributions obtained in parts (a) and (b) are of the same order of magnitude?

NOTE: The rest-mass of the electron is $m_e c^2 \simeq 5.11 \times 10^5$ eV. Other useful numbers are:

$$\alpha = \frac{e^2}{\hbar c} \simeq \frac{1}{137}\,, \qquad \mu_B = \frac{e\hbar}{2m_e c} = 5.788 \times 10^{-9}\ \text{eV/gauss}\,.$$

(d) In the ground state of helium, the total L_z and S_z vanishes. Hence, at leading order *only* the quadratic Zeeman effect is relevant. Compute the diamagnetic susceptibility of the helium atom in its ground state, and compare with the measured value of -1.88×10^{-6} cm^3/mole.

HINT: For the ground state helium wave function, use the wave function obtained in class by the variational method (recall that the result was a product of two ground-state hydrogen atom wave functions, $\psi(\vec{\boldsymbol{r}}_1, \vec{\boldsymbol{r}}_2) = \psi_{100}(\vec{\boldsymbol{r}}_1)\psi_{100}(\vec{\boldsymbol{r}}_2)$, where

$$\psi_{100}(\vec{\boldsymbol{r}}) = \left(\frac{Z^3}{\pi a_0^3}\right)^{1/2} e^{-Zr/a_0}\,,$$

with $Z_{\rm eff} = 27/16$). Thus, with hardly any additional calculation, one can write down the expression for ΔE by inspection using the results of part (a). Finally, recall that one mole consists of 6.022×10^{23} helium atoms.

3. [20] Calculate the wavelength, in centimeters, of a photon emitted under a hyperfine transition in the ground state of deuterium. Deuterium is "heavy" hydrogen, with an extra neutron in the nucleus. The proton and neutron bind together to form a *deuteron*, with spin 1 and magnetic moment

$$\vec{\boldsymbol{\mu}}_d = \frac{g_d e}{2M_d}\,\vec{\boldsymbol{I}}\,,$$

where $\vec{\boldsymbol{I}}$ is the spin-vector of the deuteron, $g_d = 1.71$ is the deuteron g-factor and M_d is the mass of the deuteron.

<u>*DATA:*</u> Take the deuteron mass to be roughly twice the proton mass. You can also use the following results. The ratio of the proton to electron mass is $M_p/m_e \simeq 1836$. The rest-mass of the electron is $m_e c^2 \simeq 5.11 \times 10^5$ eV. Other useful numbers are:

$$\alpha = \frac{e^2}{\hbar c} \simeq \frac{1}{137}\,, \qquad \hbar c = 1.973\times 10^{-5}\ \text{eV-cm}\,, \qquad a_0 = \frac{\hbar^2}{me^2} = \frac{\hbar}{mc\alpha} \simeq 0.5\times 10^{-8}\ \text{cm}\,.$$