Airy functions and their properties

Airy functions are defined as solutions to the following differential equation:

$$\frac{d^2\psi}{dz^2} - z\psi(z) = 0\,.$$

The two independent solutions are the Airy functions of the first and second kind:

$$\psi(z) = C_1\mathrm{Ai}(z) + C_2\mathrm{Bi}(z)\,,$$

where C_1 and C_2 are constant coefficients to be determined by the boundary conditions of the problem. The Airy functions are related to Bessel functions of one-third order:

$$\begin{aligned}
\mathrm{Ai}(z) &= \frac{z^{1/2}}{3}\left[I_{-1/3}\left(\frac{2z^{3/2}}{3}\right) - I_{1/3}\left(\frac{2z^{3/2}}{3}\right)\right]\,, \\
\mathrm{Bi}(z) &= \left(\frac{z}{3}\right)^{1/2}\left[I_{-1/3}\left(\frac{2z^{3/2}}{3}\right) + I_{1/3}\left(\frac{2z^{3/2}}{3}\right)\right]\,,
\end{aligned}$$

where $I_\nu(z)$ is the Bessel function of imaginary argument.

The most important properties of the Airy functions for our purposes are the following asymptotic expansions, valid in the limit of real $z \to \infty$:

$$\begin{aligned}
\mathrm{Ai}(z) &\sim \frac{1}{2\sqrt{\pi}}\, z^{-1/4}\, \exp\left(-\frac{2}{3}z^{3/2}\right) \\
\mathrm{Ai}(-z) &\sim \frac{1}{\sqrt{\pi}}\, z^{-1/4}\, \cos\left(\frac{2}{3}z^{3/2} - \frac{\pi}{4}\right) \\
\mathrm{Bi}(z) &\sim \frac{1}{\sqrt{\pi}}\, z^{-1/4}\, \exp\left(+\frac{2}{3}z^{3/2}\right) \\
\mathrm{Bi}(-z) &\sim \frac{-1}{\sqrt{\pi}}\, z^{-1/4}\, \sin\left(\frac{2}{3}z^{3/2} - \frac{\pi}{4}\right)
\end{aligned}$$

REFERENCE:

N.N. Lebedev, *Special Functions and Their Applications* (Dover Publications Inc., Mineola, NY, 1972).