A New Rigorous Approach to Coulomb Scattering.

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Summary. — This paper establishes two properties of nonrelativistic Coulomb scattering. The first is that, when considered as a distribution, the Coulomb partial-wave series is convergent (even though divergent as a function) and converges to the Coulomb amplitude. The second property, the proof of which uses the first, is that the amplitude for any screened Coulomb potential converges as a distribution to the Coulomb amplitude (times an overall phase factor) when the screening radius tends to infinity. It is argued that this second property can be made the basis of an economical but rigorous theory of Coulomb scattering.

1. – Introduction.

One of the most famous and frustrating anomalies in scattering theory is Coulomb scattering. Because the Coulomb potential

$$V(r) = \frac{\gamma}{r}$$

falls off so slowly when $r \to \infty$, almost none of the standard results of ordinary nonrelativistic scattering can be applied in the case of Coulomb scattering. To mention just four such results, we remind the reader that 1) the asymptotic

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condition—the cornerstone of conventional scattering theory— is not satisfied when Coulomb forces are present; 2) with a Coulomb potential, the standard definitions of the scattering amplitude and phase shifts are meaningless; 3) the Coulomb partial-wave series is divergent; 4) the amplitude for a screened Coulomb potential has no limit when the screening radius goes to infinity.

One way of avoiding some of the difficulties has been to broaden the normal framework of scattering theory so as to include the anomalous case of the Coulomb potential. This approach has been pursued by DOLLARD ([1]) and by AMREIN, MARTIN and MISRA ([2]), who have been able to replace the usual asymptotic condition with a weaker, more general condition that *is* satisfied by the Coulomb potential. Since their generalized asymptotic condition is still strong enough to describe the essential observed features of scattering experiments, their work provides a rigorous and physically satisfactory basis for a theory that can handle problems involving Coulomb forces.

Nevertheless, one cannot help feeling that the labour of constructing the theory of DOLLARD *et al.* is, in a certain sense, unnecessary. This feeling is based on the fact that in practice Coulomb potentials are always screened. (For example, the Coulomb potential of a nucleus is screened by the atomic electrons, and even in the best available vacuum the potential of an isolated charge would be screened by polarization of the residual particles.) This fact suggests that one ought to be able to build a theory of Coulomb scattering which discusses only screened Coulomb potentials. Since these screened potentials are « well behaved » short-range potentials, no extension of the conventional (short-range) theory would be needed.

In this paper I propose a theory of Coulomb scattering on these lines. The theory is based on two main results, which are as follows.

The first result concerns the Coulomb partial-wave series. It has long been known (see ref. ([3])) that the partial-wave series constructed from the Coulomb partial-wave amplitude is divergent, and conversely that the full Coulomb amplitude has no partial-wave expansion. (We review both of these disagreeable results in Sect. **2**.) However, I shall prove in Sect. **3** that, *when considered as a distribution*, the Coulomb partial-wave series is convergent and that its limit (as a distribution) is precisely the full Coulomb amplitude. More explicitly, if we multiply the partial-wave series by any suitably smooth function ([*]) $\varphi(\theta)$ and integrate over all angles, then the resulting series is convergent, and

([1]) J. D. DOLLARD: *Journ. Math. Phys.*, **5**, 729 (1964).
([2]) W. O. AMREIN, P. A. MARTIN and B. MISRA: *Helv. Phys. Acta*, **43**, 313 (1970).
([3]) L. MARQUEZ: *Am. Journ. Phys.*, **40**, 1420 (1972).
([*]) As discussed in Sect. **3**, we shall require that φ vanish in the direction $\theta = 0$ and that, as a function of $\cos\theta$, it be twice differentiable with continuous second derivative.

its sum is precisely the integral of $\varphi(\theta)$ times the full Coulomb amplitude (*).

It should perhaps be emphasized that this result—that the Coulomb partial-wave series is equal to the Coulomb amplitude provided both are multiplied by $\varphi(\theta)$ and integrated first—is entirely natural and satisfactory. The point is that in any real experiment one does not observe the amplitude itself, but rather the integral of the amplitude times some smooth $\varphi(\theta)$. For example, in the scattering of a single particle off a fixed potential, the initial state of any experiment is characterized by some wave packet $\varphi_{\text{in}}(\boldsymbol{p})$; and the quantity that is observed is the corresponding outgoing packet

$$\varphi_{\text{out}}(\boldsymbol{p}) = \int \mathrm{d}^3p' \langle \boldsymbol{p}|S|\boldsymbol{p}'\rangle \varphi_{\text{in}}(\boldsymbol{p}') \propto \int \mathrm{d}\Omega' f(\boldsymbol{p} \leftarrow \boldsymbol{p}')\varphi_{\text{in}}(\boldsymbol{p}'),$$

where $\langle \boldsymbol{p}|S|\boldsymbol{p}'\rangle$ denotes the momentum-space S-matrix element and $f(\boldsymbol{p} \leftarrow \boldsymbol{p}')$ the corresponding scattering amplitude (all in the notation of ref. [5]). The most that can be measured—even in principle, and in practice we measure much less—is this integral involving the amplitude times the ingoing wave function; that is, we do not observe the amplitude itself, but only the amplitude «smeared» by some suitable test function. This means that our result —convergence of the smeared partial-wave series to the smeared amplitude— is exactly what is wanted for discussion of real measurements (**).

The second main result, whose proof uses the first, concerns the possibility of replacing the Coulomb potential

$$V(r) = \gamma/r \tag{1.1}$$

by a *screened* Coulomb potential, in the limit that the screening radius is made large. The precise nature of the screening is unimportant; for example, we could consider an exponentially screened potential

$$V^{\varrho}(r) = \frac{\gamma}{r}\exp[-r/\varrho], \tag{1.2}$$

(*) That some such result must hold has been suggested by several authors. In particular, HOLDEMAN and THALER [4] have stated precisely our result, that the Coulomb partial-wave series converges as a distribution to the Coulomb amplitude. However, they make no attempt to justify the assertion (concentrating instead on showing that the coefficients in a «formal» Legendre expansion of the Coulomb amplitude can be given a meaning) and there appears to be no published proof of the result.

[4] J. T. HOLDEMAN and R. M. THALER: *Phys. Rev.*, **139**, B 1186 (1965).
[5] J. R. TAYLOR: *Scattering Theory* (New York, N. Y., 1972).

(**) In the case of short-range potentials, it is of course very useful that the partial-wave series converges to the amplitude *as a function*, without any smearing. Nonetheless, only the weaker result (convergence as a distribution) is really needed; and in Coulomb scattering only the weaker result holds.

or a sharply truncated potential

$$V^\varrho(r) = \frac{\gamma}{r}\theta(\varrho - r)\,, \tag{1.3}$$

where $\theta(x)$ is the step function

$$\theta(x) = 0 \quad (x < 0)\,,$$
$$\qquad = 1 \quad (x \geqslant 0)\,,$$

or, quite generally,

$$V^\varrho(r) = \frac{\gamma}{r}\alpha^\varrho(r)\,, \tag{1.4}$$

where the screening function $\alpha^\varrho(r)$ tends to zero as $r \to \infty$ (with ϱ fixed) but approaches 1 as the screening radius $\varrho \to \infty$ (with r fixed) (*).

The motivation for considering such potentials has already been mentioned briefly: First, the screened potential V^ϱ is a short-range potential and so (unlike the Coulomb potential) it satisfies all of the basic assumptions of ordinary scattering theory; second, one would certainly *expect* the observable properties of V^ϱ to approach those of the Coulomb potential V as $\varrho \to \infty$. These ideas are not new. In fact they are almost as old as scattering theory itself—dating back at least to Gordon's classic paper on Coulomb scattering ([6])—and have been stated explicitly by GOLDBERGER and WATSON (ref. ([7]), pp. 259-269), for example. Nonetheless, there appears to have been no clear statement (much less a proof) of the sense in which the properties of V^ϱ go over to those of V as $\varrho \to \infty$. The two main difficulties have always been, first, that *as a function* the amplitude for V^ϱ does *not* (in general) approach the Coulomb amplitude (**), and, second, that the simplest way to approach the problem is by means of the partial-wave series, which in the Coulomb case is unfortunately divergent.

The result which I shall prove in Sect. **4** is that *as a distribution* the amplitude for the screened potential V^ϱ *does* converge to the Coulomb amplitude, times an overall phase factor. This means that the physically observable prop-

(*) Our precise assumptions about the function $\alpha^\varrho(r)$ are given in Sect. **4**.

([6]) W. GORDON: *Zeits. Phys.*, **48**, 180 (1928).

([7]) M. L. GOLDBERGER and K. M. WATSON: *Collision Theory* (New York, N. Y., 1964).

(**) As we shall discuss in Sect. **4**, there may be certain screening functions for which the screened amplitude *does* converge to the Coulomb amplitude, at least within an overall phase factor. For example, this is apparently true for the exponential screening (1.2), at least if γ is small. Nonetheless, this is certainly *not* true for all screening functions.

erties of V^ϱ become independent of ϱ as ϱ becomes large, and that in the limit they coincide precisely with the corresponding Coulomb quantities.

This result is the essential foundation of the alternative treatment of Coulomb scattering mentioned at the beginning of this Introduction. This treatment would begin with the observation that in practice the potential of any point charge is always screened; that is, the actual potential in any real Coulomb-scattering experiment is *not* the pure Coulomb (1.1) but is a screened Coulomb of the type (1.4). Since this potential is of short range all of the standard results of scattering theory hold good. The whole scattering process can therefore be satisfactorily described and the various scattering probabilities computed within the normal theoretical framework. Having carried through this whole programme, we would then appeal to our result to guarantee that everything obtained in this way is actually independent of the exact nature of the screening and of the precise value of the screening radius ϱ, provided only that ϱ is large; and that the results coincide with what one would obtain using the pure Coulomb amplitude.

To summarise briefly: Our results establish that Coulomb-scattering experiments can be consistently described within the framework of conventional scattering theory by the use of screened Coulomb potentials, and that the resulting predictions are independent of the nature and radius ϱ of the screening, provided ϱ is large. Further, they justify the use of the pure Coulomb amplitude in situations where the Coulomb potential is in fact always screened.

2. – Divergence of the Coulomb partial-wave series as a function.

In this Section we introduce the necessary notations, and review the reasons for the divergence of the Coulomb partial-wave series.

We shall discuss the scattering of a spinless particle with charge e_1 and mass m off a fixed target of charge e_2. The potential is therefore

$$V(r) = e_1 e_2 / r \equiv \gamma / r \,. \tag{2.1}$$

We use units with $\hbar = m = 1$ and, since we shall only be concerned with a single fixed energy, we shall also set the incident momentum $p = 1$. We shall throughout use the abbreviation

$$x \equiv \cos\theta \,,$$

where θ is the usual scattering angle.

It is a notorious fact that in Coulomb scattering one cannot define either a phase shift or a scattering amplitude, in the ordinary sense. Nonetheless, it is usual to introduce what are called the Coulomb phase shifts and Coulomb amplitude, which are defined as follows:

$$\sigma_l = \text{Coulomb phase shift} = \arg \Gamma(l+1+i\gamma) \tag{2.2}$$

and

$$f(x) = \text{Coulomb amplitude} = -\gamma \exp[2i\sigma_0] \frac{\exp[-i\gamma \ln \sin^2(\theta/2)]}{2\sin^2(\theta/2)} = \tag{2.3}$$

$$= \gamma' \frac{\exp[-i\gamma \ln(1-x)]}{1-x} = \gamma'(1-x)^{-1-i\gamma},$$

where $x = \cos\theta$ as usual and

$$\gamma' \equiv -\gamma \exp[2i\sigma_0] \exp[i\gamma \ln 2].$$

In terms of the Coulomb phase shifts σ_l, one defines a Coulomb partial-wave amplitude in the usual way:

$$f_l = \frac{\exp[2i\sigma_l]-1}{2i} = \frac{s_l - 1}{2i},$$

where

$$s_l = \exp[2i\sigma_l] = \frac{\Gamma(l+1+i\gamma)}{\Gamma(l+1-i\gamma)}$$

is called the Coulomb partial-wave S-matrix. While none of these quantities have their conventional significance (in terms of asymptotic forms of wave functions, for example), it is true that the experimentally observed Rutherford cross-section is correctly given by the usual formula

$$\frac{d\sigma}{d\Omega} = |f(x)|^2 = \frac{\gamma^2}{4\sin^4(\theta/2)}.$$

Before proceeding to the main results of this paper, I would like to recall two important facts. First, if one defines a Coulomb partial-wave series

$$\sum_0^\infty (2l+1) f_l P_l(x) \qquad \text{divergent}, \tag{2.4}$$

then this series is divergent; and second, it is impossible to expand the full Coulomb amplitude in a Legendre series of the type

$$f(x) = \Sigma(2l+1) a_l P_l(x) \qquad \text{false}. \tag{2.5}$$

To see that the partial-wave series (2.4) is divergent we note, from their definition (2.2), that the phase shifts σ_l satisfy

$$\sigma_l = \sigma_{l-1} + \operatorname{arctg}(\gamma/l) \xrightarrow[l\to\infty]{} \sigma_{l-1} + \gamma/l \,.$$

Thus as $l \to \infty$, the phase shifts do not go to zero; instead they increase without limit like $\ln l$. This means that the partial-wave amplitudes do not tend to zero, but move indefinitely around the unitary circle, approaching their maximum value (when $f_l = i$) at regular intervals. Since

$$P_l(\pm 1) = (\pm 1)^l \,,$$

it is immediately clear that the partial-wave series (2.4) is divergent in the forward and backward directions. To find out what happens at other angles we need Laplace's formula (see, for example, ref. [8], p. 193)

$$P_l(\cos\theta) = \left(\frac{2}{\pi l \sin\theta}\right)^{\frac{1}{2}} \cos\left(\left(l+\frac{1}{2}\right)\theta - \frac{\pi}{4}\right) + O(l^{-\frac{3}{2}})\,, \tag{2.6}$$

which holds uniformly on any interval

$$-1 + \varepsilon \leqslant \cos\theta \leqslant 1 - \varepsilon$$

($\varepsilon > 0$) as $l \to \infty$. The decrease, like $l^{-\frac{1}{2}}$, of the Legendre polynomials is clearly swamped by the factor $2l+1$ in the partial-wave series, the terms of the series oscillate under an envelope proportional to $l^{\frac{1}{2}}$, and the series diverges.

We remark that the divergence of the Coulomb partial-wave series in the forward direction ($\theta = 0$) is certainly to be expected, since the Coulomb amplitude is itself infinite at $\theta = 0$. However, the divergence for $\theta \neq 0$ is perhaps unexpected and certainly a nuisance, since when $\theta \neq 0$ the Coulomb amplitude and cross-section are finite and completely well behaved. In particular, since the partial-wave series constructed with the phase shifts σ_l is divergent, it becomes a little obscure why the σ_l should be called « Coulomb phase shifts » at all.

To see that the full Coulomb amplitude $f(x)$ cannot have any partial-wave expansion of the type (2.5), let us assume that such an expansion is possible. To simplify the argument we shall suppose that the expansion is uniformly convergent, in which case we shall arrive at an immediate contradiction. This argument will leave open the relatively exotic possibility that an expansion like (2.5) exists but is not uniformly convergent; the proof that this too is impossible is more complicated and will be given later, at the end of Sect. **5**.

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[8] G. Szegö: *Orthogonal Polynomials* (New York, N. Y., 1959).

If we assume that the expansion (2.5) is uniformly convergent for all x, then it follows that the expansion coefficients are given by the familiar formula

$$a_l = \frac{1}{2}\int_{-1}^{1} f(x) P_l(x)\,\mathrm{d}x\,. \tag{2.7}$$

The integrand $f(x)P_l(x)$ is continuous and bounded on any subinterval $[-1, 1-\varepsilon]$; thus the integral

$$a_l(\varepsilon) = \frac{1}{2}\int_{-1}^{1-\varepsilon} f(x) P_l(x)\,\mathrm{d}x \tag{2.8}$$

certainly exists for any $\varepsilon > 0$. Unfortunately the singularity of $f(x)$ at $x = 1$ means that this integral has no limit as $\varepsilon \to 0$. To prove this it is sufficient to consider the case $l = 0$ (since all of the P_l approach 1 at $x = 1$)

$$a_0(\varepsilon) = \frac{1}{2}\int_{-1}^{1-\varepsilon} f(x)\,\mathrm{d}x = (\gamma'/2)\int_{-1}^{1-\varepsilon} \exp[-i\gamma \ln(1-x)]\,\mathrm{d}x/(1-x) =$$

$$= (\gamma'/2)\int_{-\ln 2}^{-\ln\varepsilon} \exp[i\gamma\xi]\,\mathrm{d}\xi\,.$$

As $\varepsilon \to 0$, the upper limit tends to infinity and the integral oscillates without limit. (For future reference, we note, however, that $a_l(\varepsilon)$ does remain *bounded* as $\varepsilon \to 0$.)

Since $a_l(\varepsilon)$ has no limit as $\varepsilon \to 0$, it follows that the original integral (2.7) for a_l cannot exist, and we have the desired contradiction.

3. – Convergence of the Coulomb partial-wave series as a distribution.

We shall now show that, even though the Coulomb partial-wave series is divergent as a function, it is nonetheless convergent as a distribution, and that its limit is the full Coulomb amplitude. We shall write this result as

$$\sum_{0}^{\infty} (2l+1) f_l\, P_l(x) = f(x) \quad \text{(as a distribution)} \tag{3.1}$$

to emphasize that the limit is in the sense of distributions. This means that, if we multiply both sides by a suitably smooth test function $\varphi(x)$ and integrate from -1 to 1, then the resulting two numbers are equal:

$$\sum_0^\infty \left\{ \int_{-1}^{1} \varphi(x)(2l+1) f_l P_l(x)\,dx \right\} = \int_{-1}^{1} \varphi(x) f(x)\,dx\,. \tag{3.2}$$

It is convenient to introduce the notation

$$\varphi_l \equiv (2l+1) \int_{-1}^{1} \varphi(x) P_l(x)\,dx\,, \tag{3.3}$$

in terms of which the desired result becomes

$$\sum_0^\infty \varphi_l f_l = \int_{-1}^{1} \varphi(x) f(x)\,dx\,. \tag{3.4}$$

Naturally we expect this result to hold only when $\varphi(x)$ is in some space of « reasonable » functions. Firstly, φ must certainly be a smooth function. In this connection it would seem quite reasonable to require φ to be infinitely differentiable. However, it proves sufficient for our purposes to require only that φ be twice differentiable, with a continuous second derivative, on the closed interval $[-1, 1]$ We denote the space of all such functions by $C^2[-1, 1]$, and shall insist from now on that φ belong to this space.

Secondly, we must require that $\varphi(x)$ vanish at $x = 1$, that is at $\theta = 0$. Mathematically, this requirement is made to ensure that the integral $\int_{-1}^{1} \varphi f\,dx$ exists, inspite of the singularity of f at $x = 1$. Physically, the requirement corresponds to the well-known fact that in Coulomb scattering it is impossible to measure a meaningful forward cross-section.

Accordingly, what we shall prove is that (3.4) holds for all $\varphi(x)$ satisfying

$$\varphi \in C^2[-1, 1] \qquad \text{and} \qquad \varphi(1) = 0\,.$$

Before we prove this result we recall the important relation (*)

$$\sum (2l+1)\, P_l(x) = 2\delta(1-x) \qquad \text{(as a distribution)}\,. \tag{3.5}$$

This is a special case of the familiar completeness relation

$$\sum (2l+1) P_l(x) P_l(y) = 2\delta(x-y) \qquad \text{(as a distribution)}\,,$$

(*) In this relation the meaning of $\delta(1-x)$ is that $\int_{-1}^{1} \varphi(x)\,\delta(1-x)\,dx = \varphi(1)$.

whose proof can be found in the standard texts (*e.g.* ref. (9), pp. 425-429), and certainly holds for all test functions in $C^2[-1,1]$. In particular, we are interested in test functions φ which vanish at $x=1$. For such functions, the delta-function on the right-hand side of (3.5) produces $\varphi(1)=0$. Thus we can write

$$\sum(2l+1)P_l(x)=0 \qquad \text{(as a distribution for test functions with } \varphi(1)=0). \tag{3.6}$$

The significance of (3.6) can be seen if we note that the series on the left is a partial-wave series in which the partial-wave amplitude is a constant (*i.e.* independent of l). Thus (3.6) implies that, if two sets of partial-wave amplitudes f_l and f'_l differ by a constant

$$f'_l = f_l + \text{constant},$$

then the corresponding partial-wave series have the same sum (regarded, of course, as distributions for test functions with $\varphi(1)=0$).

We can rewrite (3.5) and (3.6) explicitly using the notation (3.3) as

$$\sum \varphi_l = 2\varphi(1) = 0 \qquad (\text{for } \varphi(1)=0)\,. \tag{3.7}$$

We begin the main proof by showing that, when smeared with a test function φ, the Coulomb partial-wave series is convergent. The series in question (after smearing) is

$$\sum_0^\infty \varphi_l f_l\,. \tag{3.8}$$

Now, the partial-wave amplitude is always bounded by one, $|f_l|\leqslant 1$. Meanwhile, as $l\to\infty$ the Legendre polynomials oscillate more and more rapidly, and the integral $\int \varphi P_l\,\mathrm{d}x$ goes to zero. In fact it is easily shown that

$$|\varphi_l| = (2l+1)\left|\int_{-1}^{1}\varphi(x)P_l(x)\,\mathrm{d}x\right| \leqslant K(l+1)^{-\frac{1}{2}} \tag{3.9}$$

for some constant K, provided φ is in $C^2[-1,1]$. (See, for example, ref. (9), pp. 427-428.) It immediately follows that the series (3.8) is convergent.

(9) W. KAPLAN: *Advanced Calculus* (Reading, Mass., 1952).

Having established that the smeared partial-wave series is convergent we can now show that its limit is the smeared Coulomb amplitude

$$\int_{-1}^{1} \varphi(x) f(x)\, \mathrm{d}x \,.$$

Since $\varphi(1) = 0$, this integral is convergent at $x = 1$ (where $f(x)$ is infinite) and it can therefore be rewritten as

$$\int_{-1}^{1} \varphi(x) f(x)\, \mathrm{d}x = \lim_{\varepsilon \to 0} \int_{-1}^{1-\varepsilon} \varphi(x) f(x)\, \mathrm{d}x \,. \tag{3.10}$$

We now note that since φ is in $C^2[-1, 1]$ it can be expanded in a Legendre series with coefficients $\varphi_l/2$:

$$\varphi(x) = \tfrac{1}{2} \sum_{0}^{\infty} \varphi_l P_l(x) \,. \tag{3.11}$$

This series is uniformly convergent on $[-1, 1]$. Therefore, we can multiply it by $f(x)$ and it is still uniformly convergent on any (*) $[-1, 1-\varepsilon]$. Thus we can substitute the Legendre expansion (3.11) into the integral (3.10) and interchange the sum and integral to give

$$\int_{-1}^{1} \varphi(x) f(x)\, \mathrm{d}x = \lim_{\varepsilon \to 0} \sum_{0}^{\infty} \varphi_l \frac{1}{2} \int_{-1}^{1-\varepsilon} f(x) P_l(x)\, \mathrm{d}x = \lim_{\varepsilon \to 0} \sum_{0}^{\infty} \varphi_l a_l(\varepsilon) \,, \tag{3.12}$$

where $a_l(\varepsilon)$ is the integral discussed in the last Section

$$a_l(\varepsilon) = \tfrac{1}{2} \int_{-1}^{1-\varepsilon} f(x) P_l(x)\, \mathrm{d}x \,. \tag{3.13}$$

Since we wish to prove that the expression (3.12) is equal to $\sum \varphi_l f_l$ one might hope to prove that

$$a_l(\varepsilon) \xrightarrow[\varepsilon \to 0]{} f_l \qquad \text{false} \,. \tag{3.14}$$

Unfortunately, this is certainly false since, as we saw in Sect. **2**, the coefficients $a_l(\varepsilon)$ have *no* limit as $\varepsilon \to 0$. However, it is easily shown (see Sect. **5**) that, although (3.14) is false, the difference $a_l(\varepsilon) - a_{l-1}(\varepsilon)$ does have a limit

(*) It is *not* uniformly convergent on $[-1, 1]$ because $f(x)$ blows up as $x \to 1$.

and in fact

$$a_l(\varepsilon) - a_{l-1}(\varepsilon) \to f_l - f_{l-1}\,. \tag{3.15}$$

If we replace l by $l-1$, $l-2, \ldots, 1$ in this relation and add the resulting limits, we find that

$$a_l(\varepsilon) - a_0(\varepsilon) \to f_l - f_0 \quad \text{or} \quad a_l(\varepsilon) - g(\varepsilon) \xrightarrow[\varepsilon\to 0]{} f_l\,, \tag{3.16}$$

where the function

$$g(\varepsilon) = a_0(\varepsilon) - f_0$$

is independent of l and bounded. That is, although $a_l(\varepsilon)$ does not approach f_l, the difference between the two does become independent of l as $\varepsilon \to 0$. Now we have seen that, if two sets of partial-wave amplitudes differ by an amount independent of l, then their partial-wave series have the same sum (as distributions); thus (3.16) implies the required result.

To make this last point precise we subtract from (3.12) the identity

$$0 = g(\varepsilon) \sum \varphi_l$$

to give

$$\int_{-1}^{1} \varphi(x) f(x)\, \mathrm{d}x = \lim_{\varepsilon\to 0} \sum_0^\infty \varphi_l\big(a_l(\varepsilon) - g(\varepsilon)\big)\,. \tag{3.17}$$

Now we shall show in Sect. **5** that

$$|a_l(\varepsilon)| < K \ln(l+2) \qquad \text{(all } l, \text{ all } \varepsilon)\,. \tag{3.18}$$

Since $g(\varepsilon)$ is bounded and $|\varphi_l| < K(l+1)^{-\frac{3}{2}}$, the series (3.17) is uniformly convergent for all ε. We can therefore interchange the limit and sum to give

$$\int_{-1}^{1} \varphi(x) f(x)\, \mathrm{d}x = \sum_0^\infty \varphi_l \lim_{\varepsilon\to 0} \big(a_l(\varepsilon) - g(\varepsilon)\big) = \sum \varphi_l f_l$$

by (3.16), as required.

In Sect. **4** we shall use this result to prove our other main result. To conclude this Section we remark that the convergence (as a distribution) of the Coulomb partial-wave series to the Coulomb amplitude establishes the sense in which the σ_l are the « correct » Coulomb phase shifts. The result also explains, to a great extent, the surprising success of the many authors who have used the series with a cavalier disregard for its divergence in the ordinary, functional, sense (see ref. [3]).

4. – The Coulomb amplitude as the limit of a screened Coulomb.

In this Section we use the result of Sect. **3** to prove our second main result. We consider a screened Coulomb potential

$$V^{\varrho}(r) = \frac{\gamma}{r}\,\alpha^{\varrho}(r)\,, \tag{4.1}$$

where the screening function $\alpha^{\varrho}(r)$ satisfies

$$0 \leqslant \alpha^{\varrho}(r) \leqslant 1\,,$$

and the two conditions

1) with ϱ fixed, $\alpha^{\varrho}(r) \to 0$ monotonically and is $O(r^{-2-\varepsilon})$ as $r\to\infty$ (some $\varepsilon > 0$);

2) with r fixed, $\alpha^{\varrho}(r) \to 1$ as $\varrho\to\infty$.

Here the condition 1) guarantees that the screened potential V^{ϱ} is a « well behaved », short-range potential. (That α^{ϱ} is monotonic is a technical assumption, which could be somewhat relaxed but is anyway quite harmless.) Condition 2) ensures that V^{ϱ} approaches the pure Coulomb potential $V = \gamma/r$ as the screening radius ϱ tends to infinity.

Subject to these conditions, we shall show that as $\varrho\to\infty$ the amplitude $f^{\varrho}(x)$ for the screened potential converges as a distribution to the Coulomb amplitude $f(x)$, times an overall phase factor. Specifically, if $\zeta(\varrho)$ denotes the real function

$$\zeta(\varrho) = -\int\limits_{\frac{1}{2}}^{\infty} V^{\varrho}(r)\,\mathrm{d}r\,, \tag{4.2}$$

then we shall show that

$$f^{\varrho}(x) \xrightarrow[\varrho\to\infty]{} \exp[2i\zeta(\varrho)]f(x) \qquad \text{(as a distribution)}\,, \tag{4.3}$$

that is, for every $\varphi(x)$ in $C^2[-1, 1]$ with $\varphi(1) = 0$,

$$\int\limits_{-1}^{1}\varphi(x)f^{\varrho}(x)\,\mathrm{d}x \xrightarrow[\varrho\to\infty]{} \exp[2i\zeta(\varrho)]\int\limits_{-1}^{1}\varphi(x)f(x)\,\mathrm{d}x\,. \tag{4.4}$$

As discussed in the Introduction, this result means that the observed outgoing wave packet scattered off V^ϱ is indistinguishable from that computed with the Coulomb amplitude $f(x)$ provided ϱ is large. This allows us to use screened potentials to build up a theory of Coulomb scattering, and justifies the use of the pure Coulomb amplitude in practical situations, where the Coulomb potential is in fact always screened.

Before going on to the proof of (4.4) it is natural to enquire whether the limit might not also hold in the ordinary (functional) sense

$$f^\varrho(x) \xrightarrow[\varrho\to\infty]{} \exp[2i\zeta(\varrho)]f(x) \tag{4.5}$$

for each x. It turns out that this depends on the nature of the screening. WEINBERG ([10]) has proved the conjecture of DALITZ ([11]) that, if γ is small enough for the Born series to converge, then for the special case of exponential screening

$$\alpha^\varrho(r) = \exp[-r/\varrho] \tag{4.6}$$

the limit (4.5) does hold. To see that in general such a limit does *not* hold, let us consider a sharp cut-off

$$\alpha^\varrho(r) = \theta(\varrho - r)\,, \tag{4.7}$$

and suppose that γ is so small that both f^ϱ and f are given by their Born approximations. In this case

$$f(x) = \frac{\gamma}{x-1} \qquad \text{and} \qquad f^\varrho(x) = \frac{\gamma}{x-1}\left(1+\cos\varrho\sqrt{2-2x}\right).$$

Obviously $f^\varrho(x)$ does not approach $f(x)$ as $\varrho\to\infty$. However, if it is first smeared with the smooth function $\varphi(x)$, then the oscillatory cosine term goes to zero as $\varrho\to\infty$; and the two amplitudes become equal as distributions.

Obviously the fact that the limit (4.5) holds as an ordinary, functional limit for the exponential screening (4.6) (if γ is small, at least) makes this particular screening function especially important in practice. Nonetheless, for our present purposes the more important result is that the limit (4.3) holds as a distribution for *all* screening functions.

The proof of (4.3) or (4.4) uses the partial-wave series for $f^\varrho(x)$ and $f(x)$. Since V^ϱ is $O(r^{-3-\varepsilon})$ as $r\to\infty$, $f^\varrho(x)$ has a partial-wave expansion

$$f^\varrho(x) = \sum_0^\infty (2l+1) f_l^\varrho P_l(x)\,,$$

([10]) S. WEINBERG: *Phys. Rev.*, **140**, B 516 (1965).
([11]) R. H. DALITZ: *Proc. Roy. Soc.*, A **206**, 509 (1951).

which converges uniformly for all x (*). This can be multiplied by $\varphi(x)$ and integrated term by term, to give for the left-hand side of (4.4)

$$\int_{-1}^{1} \varphi(x) f^{\varrho}(x)\, \mathrm{d}x = \sum_{0}^{\infty} \varphi_l f_l^{\varrho} . \tag{4.8}$$

Meanwhile, by the result of Sect. **3**, the smeared amplitude on the right-hand side of (4.4) is

$$\int_{-1}^{1} \varphi(x) f(x)\, \mathrm{d}x = \sum_{0}^{\infty} \varphi_l f_l . \tag{4.9}$$

Thus all we have to do is to show that these two partial-wave series are equal, within the prescribed phase factor, as $\varrho \to \infty$.

To evaluate the partial-wave series (4.8) we need the phase shifts δ_l^{ϱ} for the screened potential V^{ϱ}. These require a rather messy calculation, which we give in Sect. **5**. However, the result of this calculation can be easily understood, as follows: we recall that the pure Coulomb potential has no phase shift in the ordinary sense; the radial-wave functions have the asymptotic form

$$\sin\left(r - \tfrac{1}{2} l\pi + \sigma_l - \gamma \ln 2r\right) + O(r^{-1}) , \tag{4.10}$$

which, instead of settling down to a fixed phase as $r \to \infty$, continues to pick up phase logarithmically. Thus, if we consider the screened potential V^{ϱ}, then there is a well-defined phase shift δ_l^{ϱ} for each fixed ϱ; but, as $\varrho \to \infty$ and V^{ϱ} goes over to the Coulomb potential, this phase shift diverges. For example, with the sharply truncated potential given by (4.7) it is easily seen from (4.10) that

$$\delta_l^{\varrho} = \sigma_l - \gamma \ln 2\varrho + O(\varrho^{-1}) , \tag{4.11}$$

and in general, as we shall prove in Sect. **5**,

$$\delta_l^{\varrho} \xrightarrow[\varrho \to \infty]{} \sigma_l + \zeta(\varrho) \tag{4.12}$$

for any screening function, where $\zeta(\varrho)$ is the real function defined in (4.2). It is easily seen that the function $\zeta(\varrho)$, and hence the phase shift, has no limit as $\varrho \to \infty$.

(*) It is not hard to prove (by means of the bound on p. 85 of ref. (12), for example) that if $V = O(r^{-3-\varepsilon})$ as $r \to \infty$ then $|f_l| < K/l^{2+\varepsilon}$.

(12) V. DE ALFARO and T. REGGE: *Potential Scattering* (Amsterdam, 1965).

Although the phase shifts δ_l^ϱ have no limit as the screening radius $\varrho \to \infty$, the asymptotic form (4.12) allows us to prove the desired result. From (4.12) it follows that the partial-wave amplitudes behave as follows:

$$f_l^\varrho = \frac{\exp[2i\delta_l^\varrho]-1}{2i} \xrightarrow[\varrho\to\infty]{} \frac{\exp[2i(\sigma_l+\zeta(\varrho))]-1}{2i} =$$

$$= \exp[2i\zeta(\varrho)]f_l + \frac{\exp[2i\zeta(\varrho)]-1}{2i} = \exp[2i\zeta(\varrho)]f_l + g(\varrho),$$

which again has no limit as $\varrho \to \infty$. However, the important point is that, apart from the expected overall phase, f_l^ϱ approaches the Coulomb partial-wave amplitude, *plus a term which is independent of l and is bounded.* This is exactly analogous to the situation in Sect. **3** (cf. eq. (3.16), where the function $g(\varepsilon)$ as $\varepsilon \to 0$ is the analogue of our present $g(\varrho)$ as $\varrho \to \infty$). Thus exactly the proof of Sect. **3** establishes that

$$\sum \varphi_l f_l^\varrho \xrightarrow[\varrho\to\infty]{} \exp[2i\zeta(\varrho)]\sum \varphi_l f_l .$$

If we insert (4.8) and (4.9), this becomes the desired result (4.4).

5. – Some proofs.

Bound on $a_l(\varepsilon)$. We have to establish the bound (3.18)

$$|a_l(\varepsilon)| < K \ln(l+2) . \tag{5.1}$$

Since each $a_l(\varepsilon)$ (with l fixed) is bounded we can exclude the values $l=0$ and 1 from our discussion. We then write

$$a_l(\varepsilon) = \frac{1}{2}\left\{\int_{-1}^{b} + \int_{b}^{1-\varepsilon}\right\} f(x) P_l(x)\,\mathrm{d}x , \tag{5.2}$$

where the value of $b = b(l)$ is chosen (*) so that $P_l(x)$ is monotonic and positive between $x = b$ and $x = 1$. To see how to do this, we note that $P_l(\cos\theta)$ is certainly monotonic and positive between $\theta = 0$ and its first zero $\theta_1(l)$. Now it can be shown that, as $l \to \infty$, $\theta_1(l) \to \varrho_1/l$, where ϱ_1 is the first zero of the Bessel function $J_0(\varrho)$ (see ref. [8], p. 193). Thus, as a function of x, $P_l(x)$ has

(*) Once b is chosen, then for certain values of ε we will have $1-\varepsilon < b$, in which case there is no need for the second term in (5.2). We shall see that this can only improve matters.

its first zero at

$$x_1(l) \to 1 - \varrho_1^2/2l^2\,,$$

and, if we take

$$b = 1 - c/l^2$$

with c sufficiently small, then $P_l(x)$ is monotonic and positive for $b \leqslant x \leqslant 1$. This lets us apply the second mean-value theorem (*) to the second term in (5.2) to give

$$(5.3)\qquad \left|\int\limits_b^{1-\varepsilon} f(x)P_l(x)\,\mathrm{d}x\right| = \left|P_l(1-\varepsilon)\int\limits_\xi^{1-\varepsilon} f(x)\,\mathrm{d}x\right| \leqslant \text{constant} \qquad (\text{some } \xi)\,.$$

Meanwhile the first term in (5.2) satisfies

$$(5.4)\qquad \left|\int\limits_{-1}^{b} f(x)P_l(x)\,\mathrm{d}x\right| \leqslant \int\limits_{-1}^{b} |f(x)|\,\mathrm{d}x = |\gamma|\int\limits_{-1}^{b} \mathrm{d}x/(1-x) =$$

$$= |\gamma|\ln\big(2/(1-b)\big) = |\gamma|\ln(2l^2/c) \leqslant \text{constant}\times\ln l\,.$$

Combining (5.3) and (5.4) in (5.2) and choosing K appropriately we obtain the desired result (5.1).

Behaviour of $a_l(\varepsilon)$ *as* $\varepsilon \to 0$. We wish to prove the result (3.15) that as $\varepsilon \to 0$

$$a_l(\varepsilon) - a_{l-1}(\varepsilon) \to f_l - f_{l-1}\,.$$

From the definition (3.13) of the $a_l(\varepsilon)$

$$a_l(\varepsilon) - a_{l-1}(\varepsilon) = \tfrac{1}{2}\int\limits_{-1}^{1-\varepsilon} f(x)\big(P_l(x) - P_{l-1}(x)\big)\,\mathrm{d}x\,.$$

This integral has a well-defined limit as $\varepsilon \to 0$, because the singularity of $f(x)$ at $x = 1$ is cancelled by the vanishing of $P_l - P_{l-1}$. Thus

$$a_l(\varepsilon) - a_{l-1}(\varepsilon) \to \frac{1}{2}\int\limits_{-1}^{1} f(P_l - P_{l-1})\,\mathrm{d}x = \frac{\gamma'}{2}\int\limits_{-1}^{1}(1-x)^{-1-i\gamma}(P_l - P_{l-1})\,\mathrm{d}x\,.$$

(*) See, for example, WHITTAKER and WATSON ([13]), p. 66.

([13]) E. T. WHITTAKER and G. N. WATSON: *Modern Analysis* (Cambridge, 1962).

This integral is easily evaluated with the help of standard integral tables. For example, GRADSHTEYN and RYZHIK ([14]), p. 797, give the integral of $(1-x)^{\sigma}P_l$ (which is convergent for $\operatorname{Re}\sigma > -1$). From this we can write down the integral of $(1-x)^{\sigma}(P_l - P_{l-1})$ and then continue analytically to $\sigma = -1 - i\gamma$. The result is

$$a_l(\varepsilon) - a_{l-1}(\varepsilon) \to \frac{1}{2i}\frac{\Gamma(l+i\gamma)}{\Gamma(l-i\gamma)}\left\{\frac{l+i\gamma}{l-i\gamma} - 1\right\} = \frac{s_l - s_{l-1}}{2i} = f_l - f_{l-1}$$

as required.

Phase shifts for the screened potential. We wish to prove the result (4.12) that, if δ_l^{ϱ} is the phase shift for the screened potential V^{ϱ}, then as the screening radius $\varrho \to \infty$

$$\delta_l^{\varrho} \to \sigma_l + \zeta(\varrho)\,, \tag{5.5}$$

where

$$\zeta(\varrho) = -\int_{\frac{1}{2}}^{\infty} V^{\varrho}(r)\,\mathrm{d}r\,.$$

Since we are interested here in a single value of the angular momentum l, we shall suppress that variable in what follows.

To prove (5.5) we use Calogero's variable-phase method ([15]). We define the *phase function* $\delta(r)$, for any potential V, to be the phase shift obtained if V is truncated at radius r. Clearly $\delta(0) = 0$, and $\delta(\infty) = \delta$, the actual phase shift for the potential V itself. The phase function for any r is then determined by the phase equation

$$\frac{\mathrm{d}\delta}{\mathrm{d}r}(r) = -2V(r)\big(\sin\delta(r)\,\hat{n}(r) + \cos\delta(r)\,\hat{j}(r)\big)^2\,,$$

where $\hat{n}$ and $\hat{j}$ are the Riccati-Neumann and Bessel functions in the notation of ref. ([5]). Replacing these by their asymptotic forms, we get

$$\frac{\mathrm{d}\delta}{\mathrm{d}r}(r) = -2V(r)\{\sin^2\big(r + \tfrac{1}{2}l\pi + \delta(r)\big) + O(r^{-1})\} = -2V(r)\big(\sin^2 x + O(r^{-1})\big)\,, \tag{5.6}$$

where

$$x \equiv r + \tfrac{1}{2}l\pi + \delta(r)\,. \tag{5.7}$$

([14]) I. S. GRADSHTEYN and I. M. RYZHIK: *Tables of Integrals, Series and Products* (New York, N. Y., 1965).

([15]) F. CALOGERO: *The Variable Phase Approach to Potential Scattering* (New York, N. Y., 1967).

We can now apply this method to the screened potential

$$V^{\varrho}(r) = \frac{\gamma}{r}\alpha^{\varrho}(r),$$

where the screening function $\alpha^{\varrho}(r)$ satisfies the conditions discussed at the beginning of Sect. 4. If we integrate (5.6) from any fixed R to ∞, we obtain

$$\delta^{\varrho} = \delta^{\varrho}(R) - 2\int_R^{\infty} V^{\varrho}(r)\sin^2 x\,\mathrm{d}r + O(R^{-1})\,. \tag{5.8}$$

The term $\sin^2 x$ in the integrand depends in a complicated way on the phase function $\delta^{\varrho}(r)$—see (5.7) above. However it turns out that in the present case the phase function varies so slowly that we can replace the term $\sin^2 x$ by its average value $\frac{1}{2}$. To see this we write $2\sin^2 x$ as $1-\cos 2x$ and examine the integral

$$\int_R^{\infty} V^{\varrho}(r)\cos 2x\,\mathrm{d}r\,. \tag{5.9}$$

Since $\mathrm{d}\delta^{\varrho}/\mathrm{d}r$ is $O(r^{-1})$, it is clear from (5.7) that x is a strictly increasing function of r (for r sufficiently large) and *vice versa*. We can then change variables from r to x and V^{ϱ} is a monotonic function of x. This allows us to apply the second mean-value theorem, and a brief calculation then shows that the integral (5.9) is $O(R^{-1})$. We can therefore rewrite (5.8) as

$$\delta^{\varrho} = \delta^{\varrho}(R) - \int_R^{\infty} V^{\varrho}(r)\,\mathrm{d}r + O(R^{-1})\,. \tag{5.10}$$

The reader can easily check that this estimate is uniform for all ϱ.

The result (5.10) shows that, to order R^{-1}, the contribution to δ^{ϱ} from radii $r > R$ is independent of l and has the advertised dependence on the potential. It remains to show that the contribution from radii $r < R$ is related to the Coulomb phase shifts in the way specified. To this end we recall that on any fixed interval $0 < r \leqslant R$ the screened potential approaches the pure Coulomb potential as $\varrho \to \infty$; to be presise

$$V^{\varrho}(r) = \frac{\gamma}{r}\alpha^{\varrho}(r),$$

and as $\varrho \to \infty$

$$\alpha^{\varrho}(r) \to 1$$

uniformly on $[0, R]$. It is fairly obvious, and easily shown, that under these conditions the phase function $\delta^\varrho(R)$ goes over to the pure Coulomb phase function

$$\delta^\varrho(R) \xrightarrow[\varrho\to\infty]{} \delta^{\text{Coul}}(R) \qquad \text{(any fixed } R\text{)} . \tag{5.11}$$

Now the pure Coulomb phase function has already been given in (4.11) and, in our present notation, is

$$\delta^{\text{Coul}}(R) = \sigma - \gamma \ln 2R + O(R^{-1}) . \tag{5.12}$$

Our proof is now almost complete. We can first choose R so large that the terms $O(R^{-1})$ in (5.10) and (5.12) are both less than any desired small amount. Having chosen R, we can find ϱ_0 such that the two sides of (5.11) differ by less than the same small amount for all $\varrho \geqslant \varrho_0$. This establishes that

$$\delta^\varrho \xrightarrow[\varrho\to\infty]{} \sigma - \gamma \ln 2R - \int\limits_R^\infty V^\varrho(r)\,\mathrm{d}r . \tag{5.13}$$

Finally, we note that as $\varrho \to \infty$

$$\int\limits_{\frac{1}{2}}^{R} V^\varrho(r)\,\mathrm{d}r \to \gamma \ln 2R .$$

Thus, the last two terms in (5.13) can be combined as $\int\limits_{\frac{1}{2}}^{\infty} V^\varrho(r)\,\mathrm{d}r$, and we have the desired result.

Nonexistence of a partial-wave series for $f(x)$. The argument given in Sect. **2** shows that the Coulomb amplitude can never be expanded in a series

$$f(x) = \sum (2l+1)\, a_l P_l(x) \tag{5.14}$$

which is uniformly convergent. There remains the possibility that there could be an expansion of this form which is not uniformly convergent. We can exclude this as follows.

Suppose that the series (5.14) is convergent for $-1 \leqslant x < 1$. (At $x = 1$ it will presumably diverge, but this makes no difference.) We can multiply this series by $(1-x)^2$ to give

$$(1-x)^2 f(x) = \sum_0^\infty (2l+1)\, a_l (1-x)^2 P_l(x) ,$$

which is convergent for $-1 \leqslant x \leqslant 1$. Using the recursion relations for the Legendre polynomials we can write $(1-x)^2 P_l$ as a linear combination of

$P_{l-2}, ..., P_{l+2}$ and hence obtain a series

$$(5.15)\qquad (1-x)^2 f(x) = \sum_0^\infty (2l+1) b_l P_l(x) \qquad (-1 \leqslant x \leqslant 1),$$

where each coefficient b_l is a certain linear combination of the a_l. Obviously if the a_l were known we could immediately determine the b_l; conversely, it is easily checked that if the b_l were known we could calculate all of the a_l, apart from two undetermined constants.

Now, the function $(1-x)^2 f(x)$ on the left-hand side of (5.15) is a well-behaved function which has a normal Legendre expansion

$$(5.16)\qquad (1-x)^2 f(x) = \sum (2l+1) c_l P_l(x) \qquad (-1 \leqslant x \leqslant 1),$$

where the coefficients c_l are given by the usual formula

$$(5.17)\qquad c_l = \frac{1}{2}\int_{-1}^{1} (1-x)^2 f(x) P_l(x)\, \mathrm{d}x = \frac{\gamma'}{2}\int_{-1}^{1} (1-x)^{1-i\gamma} P_l(x)\, \mathrm{d}x .$$

This integral is easily evaluated (see ref. [14], p. 797).

The expansions (5.15) and (5.16) are expansions of the same function, both convergent for $-1 \leqslant x \leqslant 1$. It follows (*) that the coefficients b_l are the same as the c_l, which we have just seen can be calculated from (5.17). Once we know the b_l we can, as discussed above, determine the original a_l apart from two constants. The result of this (quite lengthy) calculation is

$$(5.18)\qquad a_l = f_l + A + Bl(l+1),$$

where f_l is the Coulomb partial-wave amplitude and A and B are undetermined constants.

Since the series (5.14) is convergent (by hypothesis) it is clear that the constant B in (5.18) is zero. Thus we have established the following result: If the expansion (5.14) converges pointwise for all $x \neq 1$, then the expansion coefficients a_l must be

$$a_l = f_l + A .$$

But from the properties of f_l established in Sect. **2**, it is clear that, with

(*) This is actually a rather subtle point. The result we are using is that if $\sum \alpha_l P_l(x) = 0$ for $-1 \leqslant x \leqslant 1$, then $\alpha_l = 0$ (even when the series is not necessary uniformly convergent). I have been unable to find this result in the literature and am much indebted to Prof. J. CLUNIE of Imperial College for kindly showing me how to prove it.

$a_l = f_l + A$, the series (5.14) is *not* convergent (whatever we take for A). This is the desired contradiction.

* * *

It is a pleasure to acknowledge the help of several friends and colleagues. The main ideas of this paper were discussed at length with D. GOODMANSON. Several proofs were much simplified with the help of E. CAMPESINO-ROMEO and M. SEMON. And the theorem quoted the footnote at the end of Sect. **5** was shown to me by Prof. J. CLUNIE.

Note added in proofs.

In a recent paper ([16]) PRUGOVECKI and ZORBAS have established a connection between screened and unscreened Coulomb amplitudes, similar to that described in Sect. **4**. While our method of proof is more elementary than theirs, their method is in most respects much more general.

([16]) E. PRUGOVECKI and J. ZORBAS: *Nucl. Phys.*, **213** A, 541 (1973).

● RIASSUNTO (*)

Con questo lavoro si stabiliscono due proprietà dello scattering non relativistico di Coulomb. La prima è che, se considerata come distribuzione, la serie delle onde parziali di Coulomb è convergente (sebbene sia divergente come funzione) e converge all'ampiezza di Coulomb. La seconda proprietà, la cui dimostrazione si avvale della prima, è che l'ampiezza di ogni potenziale di Coulomb schermato converge come distribuzione all'ampiezza di Coulomb (moltiplicata per un fattore di fase onnicomprensivo) quando il raggio di schermatura tende all'infinito. Si deduce che questa seconda proprietà può essere posta alla base di una succinta ma rigorosa teoria dello scattering di Coulomb.

(*) *Traduzione a cura della Redazione.*

Новый строгий подход к кулоновскому рассеянию.

Резюме (*). — В этой статье устанавливаются два свойства нерелятивистского кулоновского рассеяния. Первое свойство состоит в том, что купоновский ряд по парциальным волнам является сходящимся и сходится к кулоновской амплитуде. В соответствии со вторым свойством, амплитуда для любого экранированного кулоновского потенциала сходится, когда радиус экранирования стремится к бесконечности. Указывается, что это второе свойство может быть использовано, как базис для экономной, но строгой теории кулоновского рассеяния.

(*) *Переведено редакцией.*