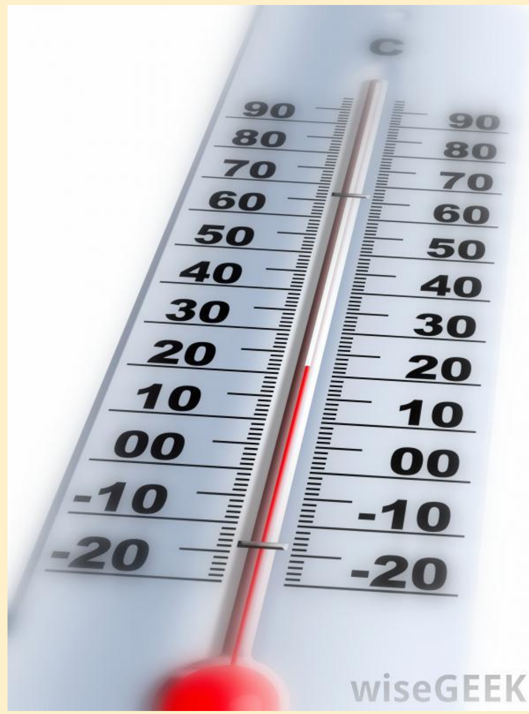
Finite Temperature Field Theory



Joe Schindler 2015

Part 1:

Basic Finite Temp Methods

Preview

"Zero Temp"	Finite Temp
Green Functions $G(x_1, x_2) = \langle 0 \vert T\phi(x_1)\phi(x_2) \vert 0\rangle$ (vacuum expectation value)	
Generating Functional for GFs $Z[J] = \int D\phi \, e^{i\int d^4x[\mathcal{L}+J\phi]}$ (real time)	

Preview

"Zero Temp"	Finite Temp
Green Functions $G(x_1, x_2) = \langle 0 \vert T\phi(x_1)\phi(x_2) \vert 0 \rangle$ (vacuum expectation value)	Green Functions $G_\beta(x_1, x_2) \propto \sum_n e^{-\beta E_n} \langle n \vert T\phi(x_1)\phi(x_2) \vert n \rangle$ (ensemble averaged expectation)
Generating Functional for GFs $Z[J] = \int D\phi \, e^{i \int d^4x [\mathcal{L} + J\phi]}$ (real time)	Generating Functional for GFs $Z[\beta, J] = \int D\phi \, e^{-\int_0^\beta d\tau \int d^3\vec{x} \, (\mathcal{L}_E + J\phi)}$ (imaginary time GFs) $Z[\beta, J] = \int D\phi_c \, e^{i \int_c dt \int d^3\vec{x} \, (\mathcal{L} + J\phi)}$ (complex time GFs)

Preview

"Zero Temp"	Finite Temp
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S-matrix and Scattering Amplitudes (from the GFs)	Not scattering amplitudes… Used more like \`correlation fns' in the stat mech sense.

Preview

"Zero Temp"

Finite Temp

In equilibrium…

Partition Function

$$Z = \mathrm{Tr}\, e^{-\beta H}$$

(canonical ensemble)

$$Z = \mathrm{Tr}\, e^{-\beta(H - \mu_i Q_i)}$$

(grand canonical)

Ensemble Average

$$\langle A \rangle = \frac{\mathrm{Tr}(A e^{-\beta H})}{\mathrm{Tr}\, e^{-\beta H}}$$

Can calculate all thermodynamic quantities from partition function.

Preview

"Zero Temp"	Finite Temp
$Z = \int D\phi \, \exp\left(i \int d^4x \, \mathcal{L}\right)$	$Z = \mathrm{Tr}\, e^{-\beta H}$
$W = -i \ln Z$	$F = -\frac{1}{\beta} \ln Z$
$\Gamma[\bar{\phi}] = W - \int J\bar{\phi}$	$\Omega(\xi) = F(\beta, \xi)$

Package deal when using Z[B,J]

Partition Function

(Canonical Ensemble)

$$Z(\beta) = \mathrm{Tr}\, e^{-\beta H}$$

Path Integral Representation

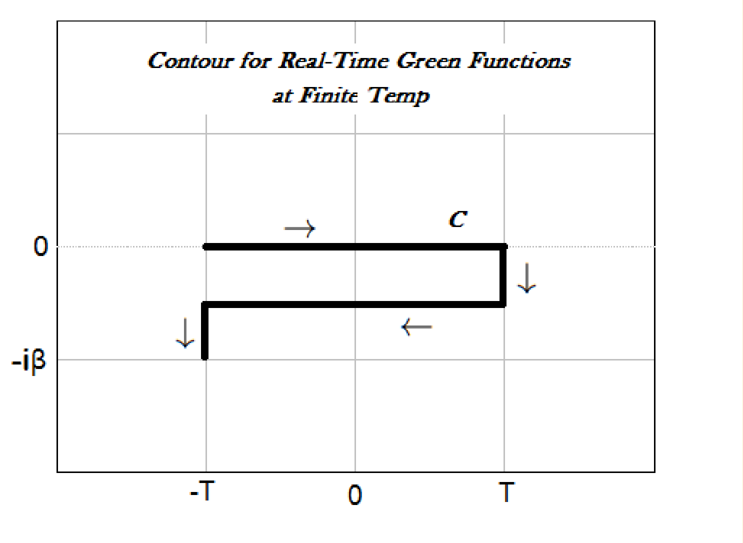
$$Z(\beta) = \int_{\phi(0)=\phi(\beta)} D\phi(\tau,\vec{x})\ \exp\left(-\int_0^\beta d\tau \int d^3\vec{x}\ \mathcal{L}_E\right)$$

Use perturbation theory and feynman diagrams to evaluate in interacting case, exactly like zero temp. How many loops?

Derive path integral rep with the identity for transition amplitudes:

$$\langle\phi_b(\vec{x})|e^{-iHT}|\phi_a(\vec{x})\rangle \propto \int_{\phi(\vec{x},0)=\phi_a(\vec{x})}^{\phi(\vec{x},T)=\phi_b(\vec{x})} D\phi(\vec{x},t)\ e^{i\int_0^T dt \int d^3\vec{x}\ \mathcal{L}(\phi)}$$

Thermal Green Functions



``Real Time" Thermal Green Functions

$$G_\beta(x_1, x_2) = \frac{1}{Z} \mathrm{Tr}\left(e^{-\beta H}\, T_c\, \phi(x_1)\phi(x_2)\right)$$

$$x_i \quad \text{on contour} \quad C$$

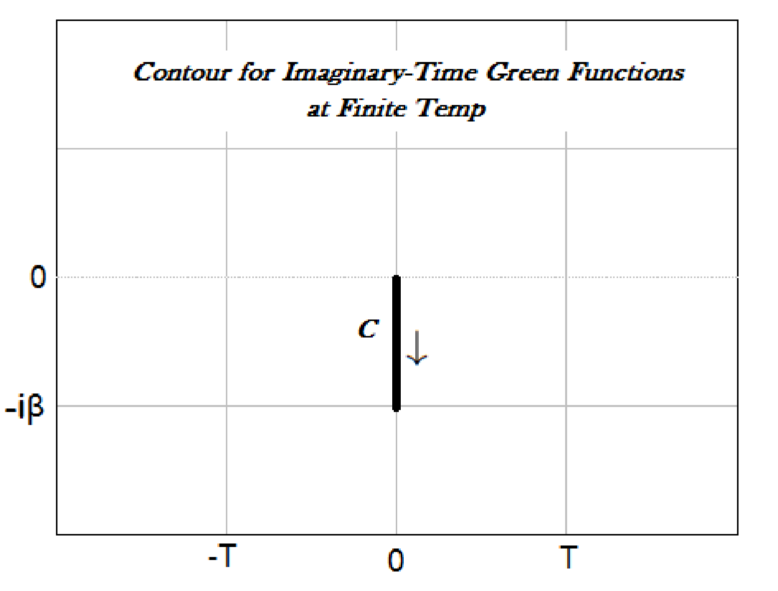
Generated by

$$Z[\beta, J] = \int D\phi_c\; e^{i \int_c dt \int d^3\vec{x}\, (\mathcal{L} + J\phi)}$$

- Can give time evolution of finite temp system away from equilibrium.
- Not necessary to assess equilibrium properties.
- Reduce to our old GFs at zero temp limit:

$$G_\beta(x_1, x_2) \propto \left(\langle 0 | T\phi(x_1)\phi(x_2) | 0 \rangle + \sum_{n=1}^{\infty} e^{-\beta E_n / E_0}\, \langle n | T\phi(x_1)\phi(x_2) | n \rangle \right)$$

Thermal Green Functions



Imaginary Time Thermal Green Functions

$$G_\beta(x_1, x_2) = \frac{1}{Z} \mathrm{Tr} \left(e^{-\beta H} \, T_c \, \phi(x_1) \phi(x_2) \right)$$

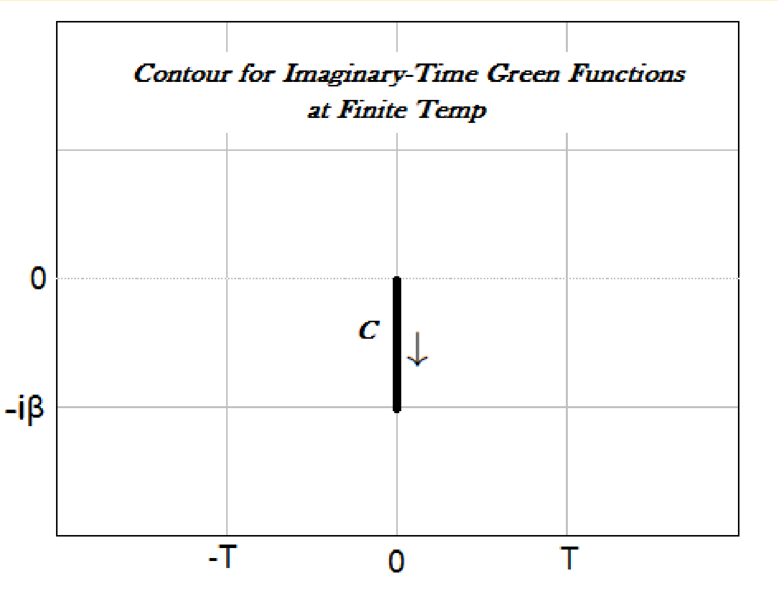
x_i on contour C

Generated by

$$Z[\beta, J] = \int D\phi \; e^{- \int_0^\beta d\tau \int d^3\vec{x} \, (\mathcal{L}_E + J\phi)}$$

- Special case of previous version, more practical for equilibrium calcs.

Thermal Green Functions



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$$G_\beta(x_1, x_2) = \frac{1}{Z} \mathrm{Tr}\left(e^{-\beta H} T_c \phi(x_1)\phi(x_2)\right)$$

$$x_i \quad \text{on contour} \quad C$$

Generated by

$$Z[\beta, J] = \int D\phi\, e^{-\int_0^\beta d\tau \int d^3\vec{x}\, (\mathcal{L}_E + J\phi)}$$

Very simple, and also gives partition function.

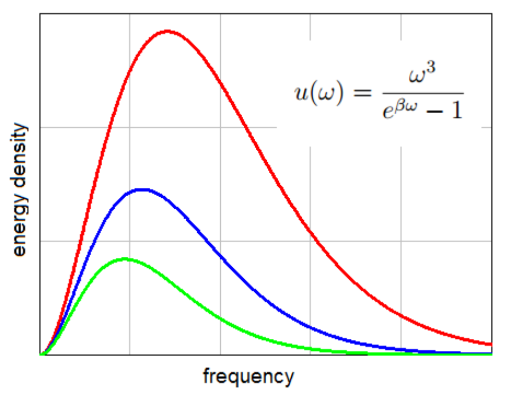
- Special case of previous version, more practical for equilibrium calcs.

relax…..

Field Thermodynamics Example

For a **free boson field** at **thermal equilibrium**, calculate energy spectrum.

$$\mathcal{L}_E = \tfrac{1}{2}\dot{\phi}^2 + \tfrac{1}{2}|\nabla\phi|^2 + \frac{1}{2}m^2\phi^2$$

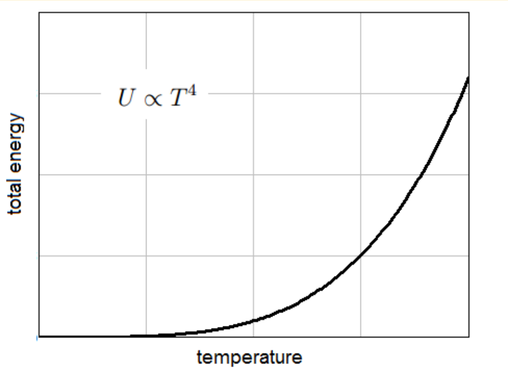


$$Z[\beta] = N(\det[-\partial_0^2 - \nabla^2 + m^2])^{-1/2}$$

$$\ln Z = V \int \frac{d^3k}{(2\pi)^3} \left(-\frac{\beta\omega_k}{2} - \ln(1 - e^{-\beta\omega_k}) \right) + const$$

$$U \equiv \frac{\langle H \rangle - E_0}{V} \qquad \langle H \rangle = -\frac{\partial \ln Z}{\partial \beta}$$

$$\xrightarrow{m=0} U \propto \int_0^\infty d\omega \, \frac{\omega^3}{e^{\beta\omega} - 1} \propto T^4$$



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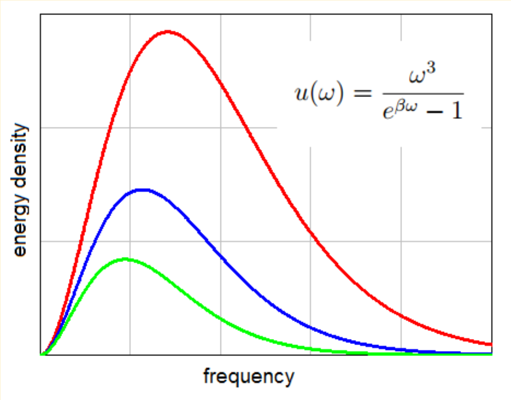
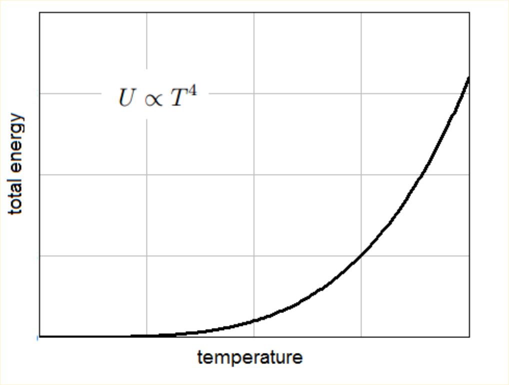


$$Z[\beta] = \int_{\phi(0)=\phi(\beta)} D\phi \ \exp \int_0^\beta d\tau \int d^3\vec{x} \ (-\mathcal{L}_E)$$

…

…

…



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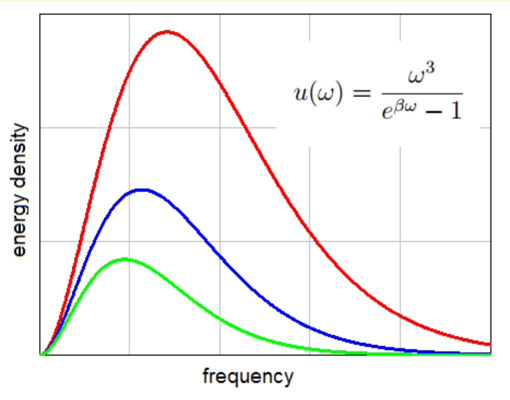

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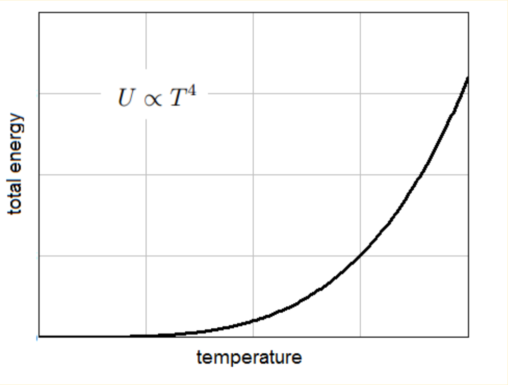
$$u(\omega) \propto \frac{\omega^3}{e^{\beta\omega} - 1}$$

$$U \propto T^4$$



Field Thermodynamics Example

$$\mathcal{L}_E = \tfrac{1}{2}\dot{\phi}^2 + \tfrac{1}{2}|\nabla\phi|^2 + \frac{1}{2}m^2\phi^2$$

Continuing with same example…

Evaluate generating functional:

$$Z[\beta, J] = N\, Z(\beta)\, \exp \frac{1}{2}\int_0^\beta d^4x d^4y\; J(x)\Delta(x-y)J(y)$$

where

$$\Delta(z) = \Delta(\tau_z, \vec{z}) = \frac{1}{\beta}\sum_{n=-\infty}^{\infty}\int \frac{d^3k}{(2\pi)^3}\, e^{i\omega_n\tau}e^{ikz}\,\Delta_n(\vec{k})$$

$$\omega_n = \frac{2\pi n}{\beta}$$

$$(-\partial_0^2 - \nabla^2 + m^2)\Delta(z) = \delta(z)$$

$$\Delta(\tau+\beta, \vec{z}) = \Delta(\tau, \vec{z})$$

$$\Delta_n(k) = \frac{1}{\omega_n^2 + \vec{k}^2 + m^n}$$

Field Thermodynamics Example

$$\mathcal{L}_E = \tfrac{1}{2}\dot{\phi}^2 + \tfrac{1}{2}|\nabla\phi|^2 + \frac{1}{2}m^2\phi^2$$

Continuing with same example…

Evaluate generating functional:

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Discrete frequency sum

$$\omega_n = \frac{2\pi n}{\beta}$$

$$(-\partial_0^2 - \nabla^2 + m^2)\Delta(z) = \delta(z)$$

$$\Delta(\tau+\beta, \vec{z}) = \Delta(\tau, \vec{z})$$

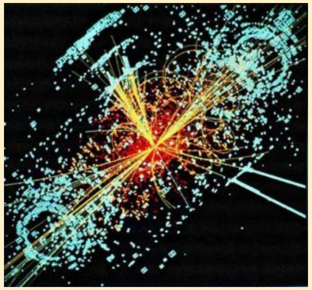
New BCs for inverse eqn

$$\Delta_n(k) = \frac{1}{\omega_n^2 + \vec{k}^2 + m^n}$$

Euclidean equivalent of minkowski propagator

Field Thermodynamics Example

Adding in Interactions??



$$\mathcal{L}_E = \tfrac{1}{2}\dot{\phi}^2 + \tfrac{1}{2}|\nabla\phi|^2 + \frac{1}{2}m^2\phi^2 + V(\phi)$$

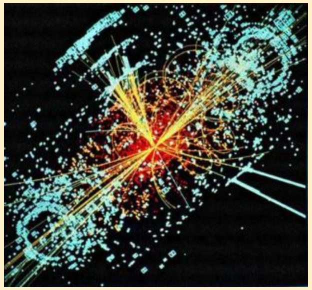
Evaluate perturbatively:

$$Z[\beta, J] = \exp\left(-\int_0^\beta d^4x\, V(-i\tfrac{\delta}{\delta J})\right)\, Z_F[\beta, J]$$

Now you have partition function including interactions to n loops!!

Field Thermodynamics Example

Adding in Interactions??



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Now you have partition function including interactions to n loops!!

-Same feynman diagram expansion as zero temp except:

- euclidean propagator
- finite tau integral / discrete frequency sum
- (for real-time formalism) thermal ghost fields

Field Thermodynamics Example

Generators and Free energies:

Partition Fn

$$Z[\beta, J] = N\, Z_0(\beta) \times \text{sum of diagrams}(\beta, J)$$
$$= N\, Z_0(\beta)\, S(\beta, J)$$

Connected Functional / Helmholtz or Gibbs FE

$$W[\beta, J] \equiv -\tfrac{1}{\beta} \ln Z[\beta, J]$$
$$= -\tfrac{1}{\beta} \ln N - \tfrac{1}{\beta} \ln Z_0(\beta) - \tfrac{1}{\beta} \ln S(\beta, J)$$
$$= W_0(\beta) - \tfrac{1}{\beta} \ln S(\beta, J)$$

$$\frac{\delta W[\beta, J]}{\delta J(x)} = \frac{1}{\beta}\, \bar{\phi}(x)$$

Effective Potential

$$\Gamma[\beta, \bar{\phi}] = W[\beta, J] - \frac{1}{\beta} \int J \bar{\phi}$$

Equilibrium Condition

$$\left.\frac{\delta \Gamma[\beta, \bar{\phi}]}{\delta \bar{\phi}}\right|_{\text{equilibrium}} = 0$$

Part 2:

Symmetry Breaking Phase Transitions

SSB at Finite Temp

Definition (SSB at Thermal Equilibrium):
A symmetry of the lagrangian is spontaneously broken when the *ensemble average* of the field multiplet doesn't respect the symmetry.

$$\langle 0|\Phi|0\rangle \rightarrow \langle \Phi \rangle$$

Typically accompanies a 2nd order phase transition
(e.g. discontinuous heat capacity)

SSB at Finite Temp

How to analyze it?

- Expect it if there is a zero-temp VEV .
- *In principle*: Calculate finite temperature quantum effective action and apply equilibrium condition.

$$\left.\frac{\delta\Gamma[\beta,\bar{\phi}]}{\delta\bar{\phi}}\right|_{\text{equilibrium}} = 0$$

-*In practice:* Becomes a simple application of thermodynamics at tree level. Add quantum corrections by summing loop diagrams in partition function.

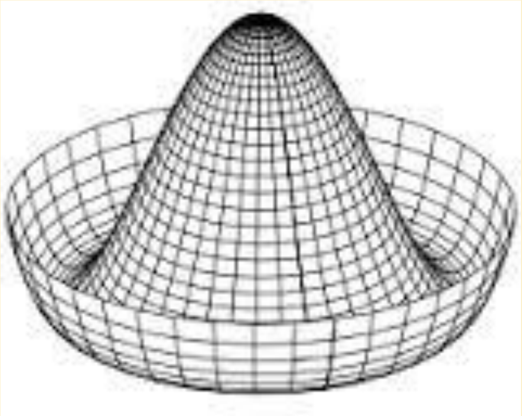
1. Expand lagrangian around the ensemble average value ξ.

2. Calculate partition function Z(β,ξ).
   (here's where quantum corrections come in)

2. Find thermodynamic effective potential Ω(β,ξ)
   (this is just free energy written as a function of ξ)
   (check: ought to obey Ω(T=0,x) = V(x))

4. Minimize effective potential wrt ξ.

SSB Example



Complex scalar with sombrero potential:

$$\mathcal{L} = \partial_\mu \phi^* \, \partial^\mu \phi - V(\phi)$$

$$V = -\mu^2(\phi^*\phi) + \lambda(\phi^*\phi)^2$$

Expand about ensemble avg:

$$\langle\phi\rangle = \xi$$
$$\phi = \tfrac{1}{\sqrt{2}}\left((\chi_1 - \sqrt{2}\,\xi) + i\chi_2\right)$$
$$\langle\chi_i\rangle = 0$$

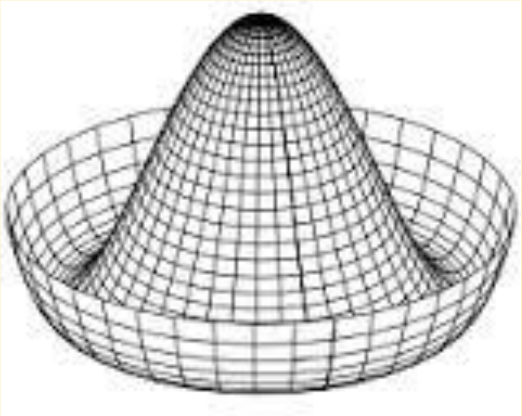
Potential becomes:

$$\begin{aligned} V &= -\mu^2(\phi^*\phi) + \lambda(\phi^*\phi)^2 \\ &= -\frac{\mu^2}{2}(\chi_1^2 + \chi_2^2 + 2\sqrt{2}\chi_1\xi + 2\xi^2) + \frac{\lambda}{4}(\chi_1^2 + \chi_2^2 + 2\sqrt{2}\chi_1\xi + 2\xi^2)^2 \\ &= \frac{1}{2}\,\chi_1^2\,(6\lambda\xi^2 - \mu^2) + \frac{1}{2}\,\chi_2^2\,(2\lambda\xi^2 - \mu^2) + U(\xi) + \mathrm{lin} + \mathrm{int} \end{aligned}$$

$$U(\xi) = -\mu^2\xi^2 + \lambda\xi^4$$

(*cheat and ignore linear term… it can be justified, see Kapusta… could avoid but we're doing simplest possible way)

SSB Example



So interesting terms are:

$$V = \tfrac{1}{2}\,\chi_1^2\,(6\lambda\xi^2 - \mu^2) + \tfrac{1}{2}\,\chi_2^2\,(2\lambda\xi^2 - \mu^2) + U(\xi) + \text{interaction}$$

Now everything is a function of ξ (and thus also of T):

$$m_1^2(\xi) = (6\lambda\xi^2 - \mu^2)$$
$$m_2^2(\xi) = (2\lambda\xi^2 - \mu^2)$$

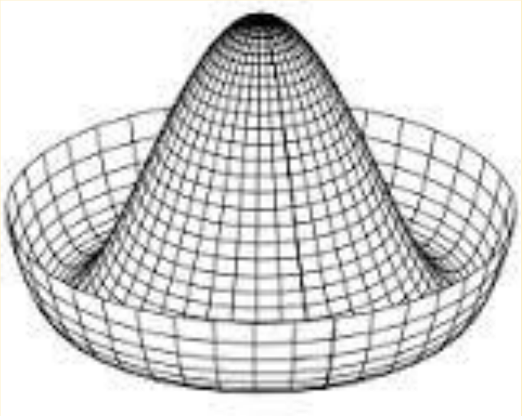
Including partition fn! Technique of making Z depend on average value is ``mean field" method.

To lowest approximation (noninteracting χ quasiparticles), the partition function factors into two noninteracting free boson field contributions.
(…this is zeroth order mean field approximation, next would be tree level mean field)

$$\begin{aligned} Z(\beta) &\propto \int D\chi_1 D\chi_2\, \exp\left(-\int_0^\beta d^4x\, \mathcal{L}_E(\chi_1,\chi_2,\xi)\right)\, e^{-\beta U(\xi)} \\ &\approx \int D\chi_1\, \exp\left(-\int \mathcal{L}_1(\chi_1,\xi)\right) \int D\chi_1\, \exp\left(-\int \mathcal{L}_1(\chi_1,\xi)\right) e^{-\beta U(\xi)} \\ &= Z_1(\beta,\xi)\, Z_2(\beta,\xi)\, e^{-\beta U(\xi)} \end{aligned}$$

Where those euclidean lagrangians are free: $\mathcal{L}_i = \tfrac{1}{2}\,\dot{\chi}_i^{\,2} + \tfrac{1}{2}\,|\nabla\chi_i|^2 + \frac{1}{2}\,m_i^2(\xi)\,\chi_i^2$

SSB Example



But we already found ln(Z) for free massive boson fields!

$$\ln Z = V \int \frac{d^3k}{(2\pi)^3} \left(-\frac{\beta\omega_k}{2} - \ln(1 - e^{-\beta\omega_k}) \right)$$

$$\omega_k^2 = k^2 + m^2$$

Thus the thermodynamic effective potential is

$$\begin{aligned}\Omega(\beta, \xi) &= -\frac{1}{\beta} \ln Z \\ &= U(\xi) - \frac{1}{\beta} \ln Z_1 - \frac{1}{\beta} \ln Z_2 \\ &= U(\xi) + \int \frac{d^3k}{(2\pi)^3} \left(\frac{\omega_1}{2} + \frac{\omega_2}{2} + \ln(1 - e^{-\beta\omega_1}) + \ln(1 - e^{-\beta\omega_2}) \right) \\ &\to U(\xi) + \int \frac{d^3k}{(2\pi)^3} \left(\ln(1 - e^{-\beta\omega_1}) + \ln(1 - e^{-\beta\omega_2}) \right)\end{aligned}$$

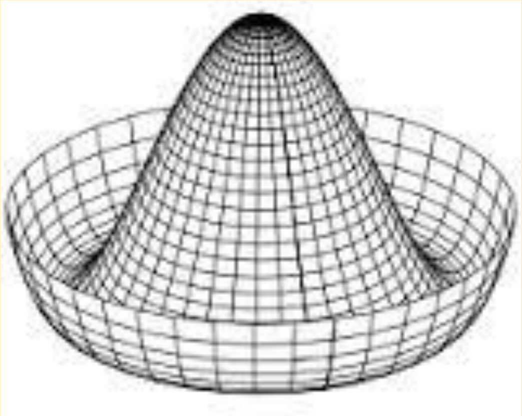
$$\omega_i = \sqrt{k^2 + m_i^2}$$

$$m_1^2(\xi) = (6\lambda\xi^2 - \mu^2)$$
$$m_2^2(\xi) = (2\lambda\xi^2 - \mu^2)$$

For temperatures $T^2 \approx \mu^2/\lambda \gg \mu^2$ can expand (see Kapusta):

$$\Omega \approx \lambda\xi^4 + (\frac{1}{3}\lambda T^2 - \mu^2)\xi^2 - \frac{\pi^2}{45}T^4 - \frac{1}{12}\mu^2 T^2$$

SSB Example



Thermodynamic potential :

$$\Omega \approx \lambda\xi^4 + (\frac{1}{3}\lambda T^2 - \mu^2)\xi^2 - \frac{\pi^2}{45}T^4 - \frac{1}{12}\mu^2 T^2$$

It's a Landau phase transition problem!
As T increases, symmetry breaking minimum disappears.

SSB occurs when

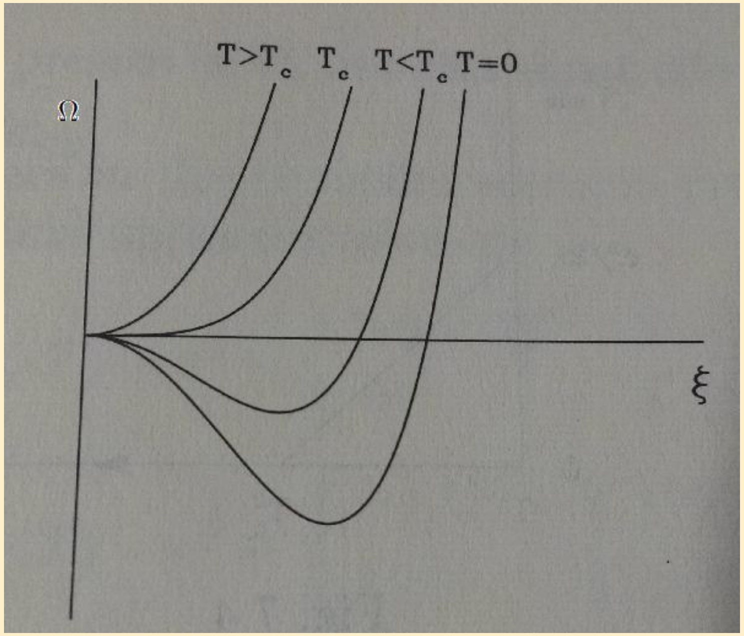
$$(\frac{1}{3}\lambda T^2 - \mu^2) < 0$$

So critical temperature is

$$T_c = 3\,\frac{\mu^2}{\lambda} = 6\xi_0$$

Noting that

$$\xi_0 = \frac{\mu^2}{2\lambda}$$



(Kapusta, pg.121)

SSB Example

At various temperatures, it acts to first approximation like two noninteracting ideal boson gases. The quasiparticle masses depend on temperature. Spontaneous symmetry breaking occurs below a critical temperature.

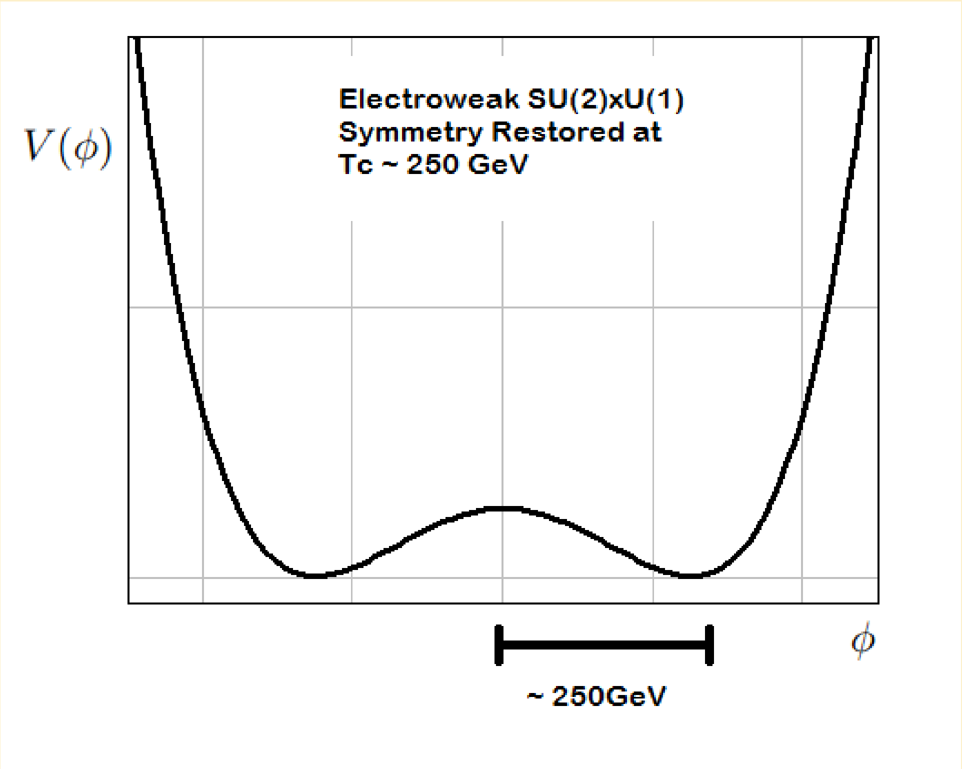
Spontaneously broken gauge theories???

(I googled "confused monkey"…)

Same analysis applies.

Electroweak Symmetry Restoration

Restored at Tc~250GeV   (see lit)



Easily estimated from Higgs potential graph.

References:

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