Anomalies

Term Project

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- Good: They are harmless and even useful when they affect global (non gauge) symmetries (Scale invariance, chirical symmetry $\pi^0 \to 2\gamma$)
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These types of anomalies (local gauge symmetries) should be canceled at all cost, otherwise the theory becomes sick, becomes inconsistent.

What miss Noether have to say?

Symmetry $\Rightarrow$

- Conservation Current

Noether's Theorem

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Noether current:

$$\partial_\mu j^\mu(x) = 0 \qquad \text{for} \qquad j^\mu(x) = \frac{\partial \mathcal{L}}{\partial(\partial\phi)})\Delta\phi - \mathcal{J}^\mu$$

Conservation Charge:

$$Q = \int d^3x \, j^0$$

QED Lagrangian:

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \overline{\psi} \left(i\partial\!\!\!/ - eA\!\!\!/ - m \right) \psi$$

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For a classical field with Lagrange density $\mathcal{L}\left(\varphi, \partial_\mu \varphi\right)$ we have that the E-L equation are

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The fermion fields ψ and $\overline{\psi} = \psi^\dagger \gamma^0$

$$\frac{\partial \mathcal{L}}{\partial \overline{\psi}} = \gamma_\mu \left(i \partial_\mu - e A_\mu \right) \psi - m \psi = 0$$

Which is the Dirac equation for ψ. For $\overline{\psi}$ we have (using $(\gamma^\mu)^\dagger = \gamma^0 \gamma^\mu \gamma^0$)

$$\left(i \partial_\mu - e A_\mu \right) \overline{\psi} \gamma_\mu - m \overline{\psi} = 0$$

Vector and axial vector currents

$$j_V^\mu = \overline{\psi}\gamma^\mu\psi \qquad j_A^\mu = \overline{\psi}\gamma^\mu\gamma^5\psi$$

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Taking the derivative of the vector current

$$\partial_\mu j_V^\mu = \partial_\mu\left(\overline{\psi}\gamma^\mu\psi\right) = \left(\partial_\mu\overline{\psi}\right)\gamma^\mu\psi + \overline{\psi}\gamma^\mu\left(\partial_\mu\psi\right)$$

And using the EOM that we found

$$\partial_\mu j_V^\mu = ieA_\mu\overline{\psi}\gamma_\mu\psi + im\overline{\psi}\psi - ieA_\mu\overline{\psi}\gamma_\mu\psi - im\overline{\psi}\psi = 0$$

Doing the same for the axial vector current

$$\begin{aligned}\partial_\mu j_A^\mu &= \partial_\mu\left(\overline{\psi}\gamma^\mu\gamma^5\psi\right) = \left(\partial_\mu\overline{\psi}\right)\gamma^\mu\gamma^5\psi - \overline{\psi}\gamma^5\gamma^\mu\left(\partial_\mu\psi\right)\\ &= ieA_\mu\overline{\psi}\gamma_\mu\gamma^5\psi + im\overline{\psi}\gamma^5\psi + ieA_\mu\overline{\psi}\gamma^5\gamma_\mu\psi + im\overline{\psi}\gamma^5\psi\\ &= 2im\overline{\psi}\gamma^5\psi\end{aligned}$$

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The divergence of the axial vector current is proportional to the fermion mass. Therefore for massless QED we have that:

$$\partial_\mu j_A^\mu = 0$$

Simple example

Decomposing the Dirac fermion into its Weyl components $\psi = \begin{pmatrix} \psi^+ \\ \psi^- \end{pmatrix}$ two currents associated with these symmetries are the vector and axial vector currents

$$j^\mu_V = \overline{\psi}\gamma^\mu\psi \qquad\qquad j^\mu_A = \overline{\psi}\gamma^\mu\gamma^5\psi$$

implies conservation of vector and axial-vector charges

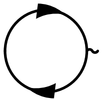
Current Matrix Elements

The expectation value of the axial current is (background electromagnetic field)

$$\langle J_A^\mu(x)\rangle_A = \frac{\int \mathcal{D}\psi\mathcal{D}\overline{\psi} J_A^\mu(x) e^{i\int d^4x\left(\overline{\psi}\partial\!\!\!/\psi - eJ_V^\mu A\mu\right)}}{\int \mathcal{D}\psi\mathcal{D}\overline{\psi} e^{i\int d^4x\left(\overline{\psi}\partial\!\!\!/\psi - eJ_V^\mu A\mu\right)}}$$

Using perturbation theory

$$\begin{aligned}\langle J_A^\mu(x)\rangle_A =& -ie\int d^4y \langle J_A^\mu(x) J_V^\alpha(y)\rangle A_\alpha(y) \\ & -\frac{e^2}{2}\int d^4y\, d^4z \langle J_A^\mu(x) J_V^\alpha(y) J_V^\beta(z)\rangle A_\alpha(y) A_\beta(z) + \ldots\end{aligned}$$



The first term is zero, because...

Loops Diagrams

Changing notation to mach Schwartz, we want to calculate calculate $\langle J_5^\alpha(x) J_\mu(y) J^\nu(z)\rangle$

$$iM^{\alpha\mu\nu} = -\int \frac{d^4k}{(2\pi)^2} Tr\left[\gamma^\mu \frac{i}{\not{k}}\gamma^\nu \frac{i}{\not{k}+\not{q}_2}\gamma^\alpha\gamma^5 \frac{i}{\not{k}-\not{q}_1} + \gamma^\nu \frac{i}{\not{k}}\gamma^\mu \frac{i}{\not{k}+\not{q}_1}\gamma^\alpha\gamma^5 \frac{i}{\not{k}-\not{q}_2}\right]$$

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$$M^{\alpha\mu\nu} = \int \frac{d^4k}{(2\pi)^2}\left[\frac{Tr\left[\gamma^\mu \not k \gamma^\nu \left(\not k + \not q_2\right)\gamma^\alpha\gamma^5\left(\not k - \not q_1\right)\right]}{k^2(k+q_2)^2(k-q_1)^2} + \frac{Tr\left[\gamma^\nu \not k \gamma^\mu \left(\not k + \not q_1\right)\gamma^\alpha\gamma^5\left(\not k - \not q_2\right)\right]}{k^2(k+q_1)^2(k-q_2)^2}\right]$$

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We can calculate our goal, which is, $p_\alpha M^{\alpha\mu\nu}$ but first lets confirm that the Ward identity works (a.k.a $q_\mu^1 M^{\alpha\mu\nu}$=0).

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$$q_\mu^1 M^{\alpha\mu\nu} = -4i\epsilon^{\alpha\nu\rho\sigma}\int \frac{d^4k}{(2\pi)^2}\left[\frac{(k-q_1)^\rho(k+q_2)^\sigma}{(k-q_1)^2(k+q_2)^2} - \frac{(k-q_2)^\rho(k+q_1)^\sigma}{(k-q_2)^2(k+q_1)^2}\right]$$

Changing variables in the first integral $k \to k + q_1$ and $k \to k + q_1$ in the second one.

Linear divergent integral

Let's consider the one-dimensional integral

$$\Delta(a) = \int_{-\infty}^{\infty} d\,x\,[f(x+a) - f(x)]$$

If the function $f(x)$ is integrable we conclude that $\Delta(a) = 0$. To see that

$$\Delta(a) = \int_{-\infty}^{\infty} d\,x \left[a f'(x) + \frac{a^2}{2} f''(x) + \ldots\right]$$

$$= a\,[f(\infty) - f(-\infty)] + \frac{a^2}{2}\,[f'(\infty) - f'(-\infty)] + \ldots$$

If the integral $\int_{-\infty}^{\infty} d\,x\,f(x)$ converges (or is at most logarithmic divergent $f(x) \approx 1/x$) then $f(\pm\infty) = f'(\pm\infty) = f''(\pm\infty) = 0$

Shifting the integration variable changes the value of a linearly divergent integral! In this case, shifting the integration variable changes the value of the integral. Such contribution is called a "surface term"

Part of the integral it is quadratically divergent, but we don't need to worry because $\epsilon^{\alpha\nu\rho\sigma}k^{\rho}k^{\sigma}=0$, then we only have a linear divergent integral (Four dimensional case).

$$\Delta^{\alpha}=\int\frac{d^4k}{(2\pi)^4}\left(F^{\alpha}[k+a]-F^{\alpha}[k]\right)$$

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$$\Delta^\alpha = \int \frac{d^4k}{(2\pi)^4}\left(F^\alpha[k+a] - F^\alpha[k]\right)$$

Doing a Wick rotation and Taylor expanding

$$\Delta^\alpha = i\int \frac{d^4k_E}{(2\pi)^4}\left\{a^\mu \frac{\partial}{\partial k_E^\mu}\left(F^\alpha[k_E]\right) + \frac{1}{2}a^\mu a^\nu \frac{\partial}{\partial k_E^\mu}\frac{\partial}{\partial k_E^\nu}\left(F^\alpha[k_E]\right)\right\}$$

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Taking the limit of $|k_E| \to \infty$ We have that :

$$\lim_{k_E \to \infty} F^\alpha[k_E] = A \frac{k_E^\alpha}{k_E^4}$$

And integrating by the surface element:

$$\Delta^\alpha = \frac{i}{32\pi^2} A a^\alpha$$

Linear divergent integrals that we shift are finite, proportional to the shift.

For the first term we have shifted k from the second by $a^\sigma = q_2^\sigma - q_1^\sigma$

$$F^{\alpha\nu} = -4i\epsilon^{\alpha\nu\rho\sigma}\frac{(q_1+q_w)^\rho k^\sigma}{(k+q_1)^2(k+q_2)^2} \longrightarrow -4i\epsilon^{\alpha\nu\rho\sigma}(q_1^\rho + q_2^\rho)\frac{k^\rho}{k^4}$$

So we get that

$$q_\mu^1 M^{\alpha\mu\nu} = \frac{1}{4\pi^2}\epsilon^{\alpha\nu\rho\sigma}q_1^\rho q_2^\sigma \neq 0$$

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We have problem!!!! The solution of this is that the equation depend of the choice of the shift of k

Dilemma

The most general shift

$$k^\mu \to k^\mu + b_1 q_1^\mu + b_2 q_2^\mu$$

This will change the results to

$$q_\mu^1 M^{\alpha\mu\nu} = \frac{1}{4\pi^2} \epsilon^{\alpha\nu\rho\sigma} q_1^\rho q_2^\sigma (1 - b_1 + b_2)$$

and for the axial current

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Thus if we take $b_1 = b_2$

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So axial current is conserved but the vector one is not.

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$$p_\alpha M^{\alpha\mu\nu} = \frac{1}{4\pi^2}\epsilon^{\mu\nu\rho\sigma} q_1^\rho q_2^\sigma = 0 \qquad q^1_\mu M^{\alpha\mu\nu} = 0$$

Dilemma

We need to choose the conservation of the vector current other wise it would be a complete disaster to lose the Ward identity. We conclude that it is impossible, as a matter of principle, not because of our limited wisdom, to simultaneously have gauge invariant and axial vector current conservation. The best choice of regulator is the one which keeps the maximum number of symmetries of the classical action.

The Axial Vector Current Operator Equation (2-d)

Peskin has a nice alternative way (Section 19.1) to derive $\partial_\mu j_5^\mu$. Defining the current by two fermion fields separate by a distance ε and the taking the limit as the two field approach each other.

$$j_5^\mu = \text{symm} \lim_{\varepsilon \to 0} \left\{ \overline{\psi}(x + \frac{\varepsilon}{2}) \gamma_\mu \gamma^5 e^{-ie \int_{x-\varepsilon/2}^{x+\varepsilon/2} dz \cdot A(z)} \psi(x - \frac{\varepsilon}{2}) \right\}$$

Where the symmetrical limit is define:

$$\text{symm} \lim_{\varepsilon \to 0} \left\{ \frac{\varepsilon^\mu}{\varepsilon^2} \right\} = 0 \qquad \text{symm} \lim_{\varepsilon \to 0} \left\{ \frac{\varepsilon^\mu \varepsilon^\nu}{\varepsilon^2} \right\} = \frac{1}{d} g^{\mu\nu}$$

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Computing the divergence of the axial current:

$$\begin{aligned}\partial_\mu j_5^\mu = \text{symm} \lim_{\varepsilon \to 0} \Bigg\{ & \left(\partial_\mu \overline{\psi}(x + \frac{\varepsilon}{2}) \right) \gamma_\mu \gamma^5 e^{-ie \int_{x-\varepsilon/2}^{x+\varepsilon/2} dz \cdot A(z)} \psi(x - \frac{\varepsilon}{2}) \\ & \overline{\psi}(x + \frac{\varepsilon}{2}) \gamma_\mu \gamma^5 e^{-ie \int_{x-\varepsilon/2}^{x+\varepsilon/2} dz \cdot A(z)} \left(\partial_\mu \psi(x - \frac{\varepsilon}{2}) \right) \\ & \overline{\psi}(x + \frac{\varepsilon}{2}) \gamma_\mu \gamma^5 \left[-i \varepsilon^\nu \partial_\mu A_\nu(x) \right] \psi(x - \frac{\varepsilon}{2}) \Bigg\}\end{aligned}$$

The Axial Vector Current Operator Equation (2-d)

Using the equation of motions:

$$\partial_\mu \overline{\psi}(x+\frac{\varepsilon}{2}) = ie\overline{\psi}(x+\frac{\varepsilon}{2})A_\mu(x+\frac{\varepsilon}{2})$$
$$\partial_\mu \psi(x-\frac{\varepsilon}{2}) = -ieA_\mu(x-\frac{\varepsilon}{2})\psi(x-\frac{\varepsilon}{2})$$

and by expanding $A_\mu(x \pm \varepsilon/2)$ and the Wilson line (Schwartz 25.45) contributions in powers of ε

$$A_\mu(x \pm \varepsilon/2) = A_\mu(x) \pm \varepsilon^\nu \partial_\nu A_\mu(x) + \ldots$$
$$e^{-ie\int_{x-\varepsilon/2}^{x+\varepsilon/2} dz^\nu A_\nu(z)} = 1 - ie\varepsilon^\nu A_\nu(x)$$
$$\partial_\mu e^{-ie\int_{x-\varepsilon/2}^{x+\varepsilon/2} dz^\nu A_\nu(z)} = -ie\varepsilon^\nu \partial_\mu A_\nu(x)$$

Summing the 3 terms:

$$\partial_\mu j_5^\mu = \text{symm} \lim_{\varepsilon\to 0} \left\{ \overline{\psi}(x+\frac{\varepsilon}{2}) \left[-ie\gamma^\mu \varepsilon^\nu (\partial_\mu A_\nu - \partial_\nu A_\mu)\right] \gamma^5 \psi(x+\frac{\varepsilon}{2}) \right\}$$

The Axial Vector Current Operator Equation (2-d)

It looks that we still get that $\partial_\mu j_5^\mu = 0$ when $\varepsilon \to 0$, but we have to be cautious because the contraction of the fermion fields.

$$\overline{\psi}(x+\frac{\varepsilon}{2})\Gamma\psi(x+\frac{\varepsilon}{2}) = \frac{-i}{2\pi}Tr\left[\frac{\gamma^\alpha\varepsilon_\alpha}{\varepsilon^2}\Gamma\right].$$

Because the contraction is singular, we have that:

$$\partial_\mu j_5^\mu = \text{symm}\lim_{\varepsilon\to 0}\left\{\frac{-i}{2\pi}Tr\left[\frac{\gamma^\alpha\varepsilon_\alpha}{\varepsilon^2}\gamma^\mu\gamma^5\right](-ie\varepsilon^\nu F_{\mu\nu})\right\}$$

In two dimensions $Tr\left[\gamma^\alpha\gamma^\mu\gamma^5\right] = 2\varepsilon^{\alpha\mu}$. Thus,

$$\partial_\mu j_5^\mu = \frac{e}{2\pi}\epsilon^{\mu\nu}F_{\mu\nu}$$

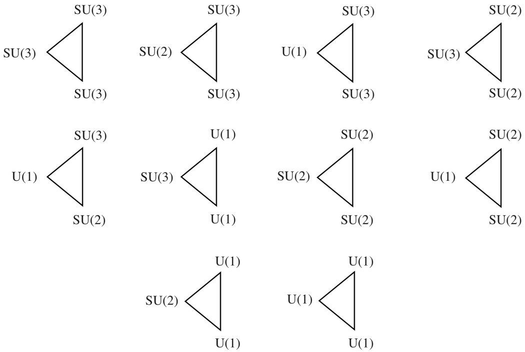
Anomaly Constrains

We want to check that the currents associated with the $SU(3)_{QCD} \times SU(2)_{\text{Weak}} \times U(1)_Y$ gauge symmetries of the SM are non-anomalous.

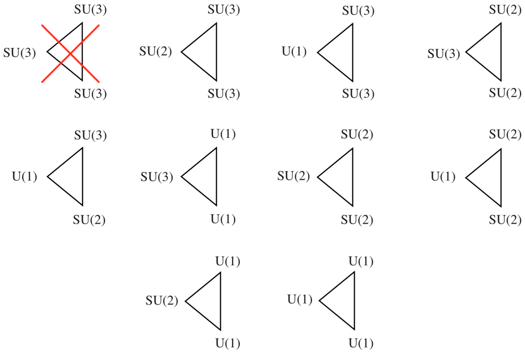
$$\partial_\alpha J^a_\alpha(x) = \left(\sum_{left} A(R_l) - \sum_{right} A(R_r) \right) \frac{g^2}{128\pi^2} d^{abc} \epsilon^{\mu\nu\alpha\beta} F^b_{\mu\nu} F^c_{\alpha\beta}$$

Where d^{abc} is define as:

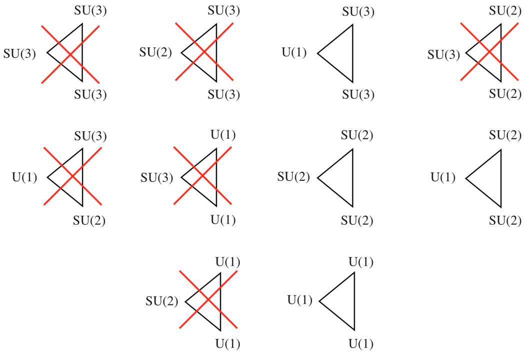
$$d^{abc} = 2Tr[T^a \{T^b, T^c\}]$$



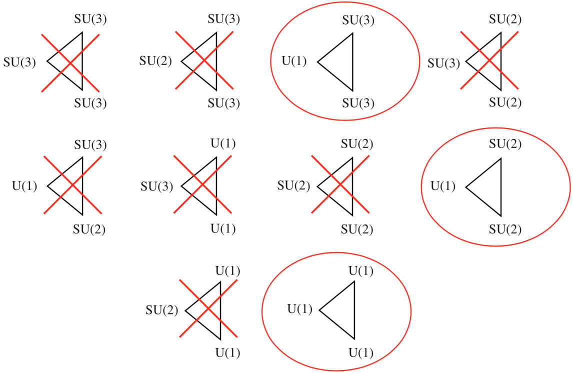
$$SU(3)_{QCD} \times SU(2)_{\text{Weak}} \times U(1)_Y$$



QCD is non-chiral, there are not $SU(3)^3$ anomalies



The generators of $SU(3)$ and $SU(2)$ are traceless



We have 3 possible anomalies

For:

$$\begin{aligned} SU(3)\times SU(3)\times U(1)_Y &\sim 2\sum Y_L - \sum Y_R \\ &= 3\times 2\times\left(\frac{1}{6}\right) - 3\times\left(\frac{2}{3}\right) - 3\times\left(-\frac{1}{3}\right) \\ &= 0 \end{aligned}$$

Lepton do not contribute. For:

$$\begin{aligned} SU(2)_{\text{Weak}}\times SU(2)_{\text{Weak}}\times U(1)_Y \sim \sum Y_L &= \text{Quarks} \quad + \quad \text{Leptons} \\ &= 3\times 2\times\left(\frac{1}{6}\right) + 2\times\left(-\frac{1}{6}\right) \\ &= 0 \end{aligned}$$

Gauge Anomaly Free

And finally:

$$U(1)_Y \times U(1)_Y \times U(1)_Y \sim$$

$$= 3 \times 2 \times \left(\frac{1}{6}\right) + 2 \times \left(-\frac{1}{2}\right)^3 - 3 \times \left(-\frac{2}{3}\right)^3 - 3 \times \left(-\frac{1}{3}\right)^3 - (-1)^3 = 0$$

So all gauge anomalies cancel!