DUE: TUESDAY, JANUARY 20, 2015

1. Show that for complex scalar fields,

$$\int \mathcal{D}\Phi^* \,\mathcal{D}\Phi \exp\left\{i\int d^4x\, d^4y\big[\Phi^*(x)M(x,y)\Phi(y)\big] + i\int d^4x\,\big[J^*(x)\Phi(x)+\Phi^*(x)J(x)\big]\right\}$$
$$= \mathcal{N}\,\frac{1}{\det M}\exp\left\{-i\int d^4x\, d^4y\, J^*(x)M^{-1}(x,y)J(y)\right\},$$

for some infinite constant $\mathcal{N}$. This is problem 14.1 on p. 283 of Schwartz.

2. (a) Derive the result:

$$\int d^4z\,\frac{\delta^2 W[J]}{\delta J(x)\delta J(z)}\frac{\delta^2\Gamma[\Phi]}{\delta\Phi(z)\delta\Phi(y)} = -\delta^4(x-y)\,,$$

and interpret diagrammatically. Here, $W[J]$ is the generating functional for the connected Green functions and $\Gamma[\Phi]$ is the generating functional for the one particle irreducible (1PI) Green functions.

(b) By taking one further functional derivative, show that Γ generates the amputated connected three-point function.

3. Consider the quantum field theory of a real scalar field governed by the Lagrangian,

$$\mathscr{L} = \tfrac{1}{2}\partial_\mu\phi\partial^\mu\phi - \tfrac{1}{2}m^2\phi^2 - \frac{\lambda}{4!}\phi^4\,.$$

(a) Evaluate the generating functional for the connected Green functions, $W[J]$, perturbatively, keeping all terms up to and including terms of $\mathcal{O}(\lambda)$.

(b) Using the result of part (a), compute the four-point connected Green function, take the appropriate Fourier transform, and verify the momentum space Feynman rule for the four-point scalar interaction obtained in class.

(c) Evaluate the classical field $\phi_c(x)$ and the generating functional for the 1PI Green functions, $\Gamma[\phi_c]$, perturbatively, keeping all terms up to and including terms of $\mathcal{O}(\lambda)$. Then, repeat part (b) for the four-point 1PI Green function.

4. Consider a scalar field theory defined by the Lagrangian density

$$\mathscr{L} = \tfrac{1}{2}\,\partial^\mu\phi(x)\partial_\mu\phi(x) - V(\phi(x))\,, \tag{1}$$

and the corresponding equation of motion,

$$\Box\,\phi(x) + V'(\phi) = 0\,,$$

where $\Box \equiv \partial^\mu\partial_\mu$ and $V' \equiv dV/d\phi$.

(a) Starting from eq. (14.122) on p. 276 of Schwartz, derive the equation of motion for the Green function $\langle\Omega|T\{\phi(x)\phi(y)\}|\Omega\rangle$,

$$\Box_x\langle\Omega|T\{\phi(x)\phi(y)\}|\Omega\rangle = -\langle\Omega|T\{V'(\phi(x))\phi(y)\}|\Omega\rangle - i\delta^4(x-y)\,. \tag{2}$$

(b) Derive eq. (2) by the following technique. Start from the path integral definition of the generating functional,

$$Z[J] = \mathcal{N}\int\mathcal{D}\phi\,\exp\left\{i\int d^4x\,\left[\mathscr{L} + J(x)\phi(x)\right]\right\}\,, \tag{3}$$

where $\mathcal{N}$ is chosen such that $Z[0]=1$. Perform a change of variables in the path integral, $\phi(x)\to\phi(x)+\varepsilon(x)$, where $\varepsilon(x)$ is an arbitrary infinitesimal function of x. Noting that a change of variables[1] does not change the value of of $Z[J]$, show that to first order in $\varepsilon(x)$,

$$\int\mathcal{D}\phi\,\exp\left\{i\int d^4x\,\left[\mathscr{L} + J(x)\phi(x)\right]\right\}\int d^4x\,\varepsilon(x)\big[-\Box\Phi - V'(\phi) + J(x)\big] = 0\,. \tag{4}$$

Since $\varepsilon(x)$ is arbitrary, we may choose $\varepsilon(x) = \epsilon\,\delta^4(x-y)$, where ϵ is an infinitesimal constant. With this choice for $\varepsilon(x)$, show that by taking the functional derivative of the eq. (4) with respect to $J(x)$ and then setting $J=0$, one can derive eq. (2).

HINT: What is the Jacobian corresponding to the change of variables, $\phi(x)\to\phi(x)+\varepsilon(x)$?

5. Consider a field theory of a real pseudoscalar field coupled to a fermion field. The interaction Lagrangian is:

$$\mathscr{L}_{\rm int} = -i\lambda\,\overline{\psi}(x)\,\gamma_5\,\psi(x)\phi(x)\,,$$

where λ is a real coupling constant (called the Yukawa coupling). Using functional techniques, derive the Feynman rule for the interaction vertex of this theory.

[1] Just as in the case of ordinary functional integration, a change of integration variables does not change the value of the functional integral.