DUE: THURSDAY, FEBRUARY 19, 2015

1. In class, I defined the matrix-valued covariant derivative operator in the adjoint representation, $\mathscr{D}_\mu$, by

$$\mathscr{D}_\mu V_\nu \equiv (D_\mu V_\nu)_a T^a = \partial_\mu V_\nu + ig[A_\mu\,,\,V_\nu]\,,$$

where $V_\nu \equiv V_\nu^a T^a$ is a matrix-valued adjoint field and $(D_\mu)_{ab} \equiv \delta_{ab}\partial_\mu + g f_{cab} A_\mu^c$ is the covariant derivative acting on a field in the adjoint representation. The commutation relations satisfied by the generators of the Lie group G are given by $[T_a\,,\,T_b] = if_{abc}T_c$, and the indices a, b and c take on d_G possible values, where d_G is the dimension of G.

(a) Prove that for any pair of matrix-valued adjoint fields V and W,

$$[\mathscr{D}_\mu\,,\,V]\,W = (\mathscr{D}_\mu V)W\,,$$

where $[\quad,\quad]$ is the usual matrix commutator. This means that $\mathscr{D}_\mu V = [\mathscr{D}_\mu\,,\,V]$ holds as an operator equation.

(b) Prove that for any matrix-valued adjoint field V,

$$[\mathscr{D}_\mu\,,\,\mathscr{D}_\nu]V = ig[F_{\mu\nu}\,,\,V]\,,$$

where $F_{\mu\nu} \equiv F_{\mu\nu}^a T^a$ is the matrix-value field strength tensor of the non-abelian gauge theory.

2. Consider the spontaneous breaking of a gauge group G down to U(1). The unbroken generator $Q = c_a T^a$ is some real linear combination of the generators of G.

(a) Prove that $x_b \equiv c_b/g_b$ is an (unnormalized) eigenvector of the vector boson squared-mass matrix, M^2_{ab}, with zero eigenvalue.

(b) Suppose that A_μ is the massless gauge field that corresponds to the generator Q. Show that the covariant derivative can be expressed in the following form:

$$D_\mu = \partial_\mu + ieQA_\mu + \ldots\,, \tag{1}$$

where we have omitted terms in eq. (1) corresponding to all the other gauge bosons and

$$e = \left[\sum_a \left(\frac{c_a}{g_a}\right)^2\right]^{-1/2}. \tag{2}$$

HINT: The vector boson mass matrix is diagonalized by an orthogonal transformation $\mathcal{O}M^2\mathcal{O}^T$ as shown in class. The rows of the matrix $\mathcal{O}$ are constructed from the *orthonormal* eigenvectors of M^2.

(c) Evaluate Q in the adjoint representation (that is, $Q = c_a T^a$, where the $(T^a)_{bc} = -if_{abc}$ are the generators of the gauge group in the adjoint representation). Show that $Q_{bc}x_c = 0$, where x_c is defined in part (a). What is the physical interpretation of this result?

(d) Prove that the commutator $[Q, M^2] = 0$, where Q is the unbroken U(1) generator in the adjoint representation and M^2 is the gauge boson squared-mass matrix. Conclude that one can always choose the eigenstates of the gauge boson squared-mass matrix to be states of definite unbroken U(1)-charge.

3. In class, we examined in detail the structure of a spontaneously broken SU(2)×U(1)$_Y$ gauge theory, in which the symmetry breaking was due to the vacuum expectation value of a $Y = 1$, SU(2) doublet of complex scalar fields. In this problem, we will replace this multiplet of scalar fields with a different representation.

(a) Consider a spontaneously broken SU(2)×U(1)$_Y$ gauge theory with a $Y = 0$, SU(2) triplet of *real* scalar fields. Assume that the electrically neutral ($Q = 0$) member of the scalar triplet acquires a vacuum expectation value (where $Q = T_3 + Y/2$). After symmetry breaking, identify the subgroup that remains unbroken. Compute the vector boson masses and the physical Higgs scalar masses in this model. Deduce the Feynman rules for the three-point interactions among the Higgs and vector bosons.

HINT: Since the triplet of scalar fields corresponds to the adjoint representation of SU(2), the corresponding SU(2) generators that act on the triplet of scalar fields can be chosen to be $(T^a)_{bc} = -i\epsilon_{abc}$. The hypercharge operator annihilates the $Y = 0$ fields. Define $L^a = ig_a T^a$, and follow the methods outlined in class.

(b) Consider a spontaneously broken SU(2)×U(1)$_Y$ gauge theory with a $Y = 2$, SU(2) triplet of *complex* scalar fields. Again, assume that the electrically neutral ($Q = 0$) member of the scalar triplet acquires a vacuum expectation value (where $Q = T_3 + Y/2$). After symmetry breaking, identify the subgroup that remains unbroken. Compute the vector boson masses and the physical Higgs scalar masses in this model.

(c) If both doublet and triplet Higgs fields exist in nature, what does this exercise imply about the parameters of the Higgs Lagrangian?

4. (a) Compute the differential cross section at $\mathcal{O}(\alpha_s^2)$ for $q\bar{q} \to t\bar{t}$ (where $q \neq t$ is any light quark and t is the top quark), in terms of the center-of-mass energy $\sqrt{s}$ and the squared four-momentum transfer t. Integrate your result over t to obtain the total cross section as a function of the squared center-of-mass energy s. In your calculation, average over initial colors and spins and sum over final colors and spins. You may assume that the initial quark and anti-quark are massless, but do *not* neglect the mass of the top-quark.

(b) Compute the differential cross section at $\mathcal{O}(\alpha_s^2)$ for $gg \to t\bar{t}$, where g is a gluon, in terms of the squared center-of-mass energy $\sqrt{s}$ and the squared four-momentum transfer t. Integrate your result over t to obtain the total cross section as a function of s. In your calculation, average over initial colors and spins and sum over final colors and spins.