DUE: TUESDAY, MARCH 17, 2015

1. By using the BRST-invariance of QED, one can derive the well known relation between the vertex function and the inverse propagator that is given in eq. (19.80) of Schwartz (on p. 352). Here is how to do it:

From part (d) of problem 1 of Problem Set 2, you know that:

$$\langle\Omega|T(\psi(x)\overline{\psi}(y)\phi(z))|\Omega\rangle = 0\,.$$

Using the BRST-invariance of the theory, this Green function must remain zero under an (infinitesimal) BRST-transformation. Computing to first order, deduce an equation that relates three different Green functions. Although some of these Green functions involve the scalar field, one may eliminate it by explicitly evaluating the scalar field propagator [after invoking one of the relations proved in part (d) of problem 1 of Problem Set 2]. Then, one can derive the Ward Identity for QED that relates the vertex function and the inverse propagator. Transforming to momentum space, check that the final result coincides with eq. (19.80) of Schwartz.

2. Consider the function of a *real* parameter z

$$F(z) \equiv \int_0^1 dx\, \ln\bigl[1 - zx(1-x) - i\epsilon\bigr]\,,$$

which appeared in the computation of the one-loop correction to the four-point function in scalar field theory.

(a) Evaluate Im $F(z)$. For what values of z does Im F vanish?

HINT: First, determine the imaginary part of the integrand. Note that $\ln\bigl[1 - zx(1-x) - i\epsilon\bigr]$ should be interpreted as the principal value of the complex-valued logarithm, with the branch cut along the negative real axis. Since ϵ is a positive infinitesimal, the sign of the imaginary part is uniquely determined. Then, carry out the integration, noting that the imaginary part of the integrand may vanish over part (and in some cases all) of the integration range.

(b) Let $\Gamma^{(4)}$ be the 1PI four-point function in a field theory of a real scalar field (with an interaction Lagrangian given by $\mathscr{L}_{\rm int} = -\lambda\phi^4/4!$). Using the *cutting rules* given in Section 24.1.2 [pp. 456–459] of Schwartz, evaluate Im $\Gamma^{(4)}$ up to order λ^2. Check your result by starting with the full $\mathcal{O}(\lambda^2)$ expression for $\Gamma^{(4)}$ obtained in class, and implementing the results of part (a).

(c) Explain briefly when you expect the evaluation of a Feynman diagram to yield non-zero imaginary part.

3. The photon vacuum polarization function is defined to be:

$$\Pi^{\mu\nu}(q) = (q^\mu q^\nu - g^{\mu\nu} q^2)\Pi(q^2)\,.$$

In class, we evaluated this function at one-loop in the $\overline{\text{MS}}$ scheme. Consider a second scheme, called the *on-shell scheme*, in which we define $\Pi(q^2 = 0) \equiv 0$.

(a) Evaluate Z_3 in this scheme.

(b) Obtain asymptotic forms for $\Pi(q^2)$ in two limiting cases: (i) $q^2 \to 0$, and (ii) $q^2 \to \infty$.

(c) Using the $q^2 \to 0$ limit of part (b), compute the $\mathcal{O}(\alpha)$ correction to the Coulomb potential. *OPTIONAL:* Compute the $\mathcal{O}(\alpha)$ correction to the Coulomb potential without making the approximation of small q^2. Examine explicitly the limiting cases $m_e r \gg 1$ and $m_e r \ll 1$.

(d) Show that the quantity:

$$\alpha_{\text{eff}}(q^2) \equiv \frac{\alpha}{1 + \Pi(q^2)}$$

is independent of whether you evaluate this expression using bare or renormalized quantities. As a result, argue that $\alpha_{\text{eff}}(q^2)$ is independent of renormalization scheme. Outline how you would relate the coupling constants defined in the $\overline{\text{MS}}$ and on-shell schemes. Sketch a graph of $\alpha_{\text{eff}}(-q^2)$ at one-loop, in the on-shell scheme, i.e. for *negative* values of the argument].

NOTE: In the on-shell scheme, $\alpha_{\text{eff}}(0)$ is the fine structure constant, which is approximately equal to $1/137$.

(e) Calculate the numerical value of the momentum scale (in GeV units) where $\alpha_{\text{eff}}(-q^2)$ blows up.

4. Consider QED coupled to a neutral scalar field:

$$\mathscr{L} = \mathscr{L}_{\text{QED}} + \tfrac{1}{2}\partial_\mu\phi\partial^\mu\phi - \tfrac{1}{2}m^2\phi^2 - \frac{\lambda}{4!}\phi^4 - g\overline{\psi}\psi\phi\,.$$

(a) Compute the amplitude for the decay $\phi \to \gamma\gamma$, as a function of m_e, m, g, and $\alpha \equiv e^2/(4\pi)$, using perturbation theory at one-loop. Simplify your answer by invoking the kinematics of the problem, i.e. momentum conservation and the on-shell conditions for the external particles. Take care to consider two diagrams which differ only in the direction of flow of electric charge in the loop. Do you need to add a counterterm in order to remove an infinity? Explain.

(b) Denote the amplitude for the scalar decay by $\mathcal{M}_{\mu\nu}$, where μ and ν are the photon Lorentz indices. Gauge invariance implies that $k_1^\mu \mathcal{M}_{\mu\nu} = k_2^\nu \mathcal{M}_{\mu\nu} = 0$, where k_1 and k_2 are the respective photon momenta. Does your amplitude of part (a) respect this requirement?

(c) Work out all integrals explicitly and evaluate the imaginary part of $\mathcal{M}_{\mu\nu}$. For what range of m_e/m is the amplitude purely real? Explain the physical significance of the non-zero imaginary part.

HINT: You may find the following integral useful:

$$\int_0^1 \frac{dy}{y} \log\bigl[1 - 4Ay(1-y)\bigr] = -2\left(\sin^{-1}\sqrt{A}\right)^2 ,$$

for $0 \leq A \leq 1$. For values of A outside this region, you may analytically continue the above result. The imaginary part of this integral is easily computed once the $i\epsilon$ factor is restored in the argument of the logarithm.

(d) Evaluate the leading behavior of $\mathcal{M}_{\mu\nu}$ in the limit of $m_e \to \infty$.

5. In QED, the renormalization group functions are:

$$\begin{aligned} \beta(e) &= \mu\frac{de_R}{d\mu} , \\ \delta(e) &= \mu\frac{da_R}{d\mu} , \\ m_R\gamma_m(e) &= \mu\frac{dm_R}{d\mu} , \\ \gamma_i(e) &= \tfrac{1}{2}\mu\frac{\partial}{\partial\mu}\ln Z_i \qquad (i = 2, 3) . \end{aligned}$$

(a) Compute $\beta(e)$, $\delta(e)$, $\gamma_m(e)$, and $\gamma_i(e)$ in the one-loop approximation, using the MS-renormalization scheme.

HINT: Most of the work has already been done for you in Section 23.2 [pp. 423–426] of Schwartz.

(b) The running coupling constant in QED can be written as:

$$\overline{\alpha}(Q) = \frac{3\pi}{\ln(\Lambda^2/Q^2)} ,$$

in the one loop approximation. Using the boundary condition $\overline{\alpha}(\mu) \equiv e_R^2/4\pi$, express Λ in terms of μ and e_R. Show that Λ is a renormalization group invariant, that is:

$$\mu\frac{d\Lambda}{d\mu} = 0 .$$

Evaluate Λ numerically.

(c) Find the relation between the $\overline{\text{MS}}$ mass parameter, m_R, and the physical electron mass m_e (i.e., the pole mass) in the one-loop approximation.