Group Structure of Spontaneously Broken Gauge Theories

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Outline

- Symmetries of the Lagrangian
  - Global symmetries
  - Spontaneous symmetry breaking of a global symmetry
  - Goldstone’s Theorem

- Gauge Theories
  - Local symmetries (a.k.a. Gauge symmetries)
  - Spontaneous symmetry breaking of a gauge symmetry: The Higgs Mechanism

- Group Structure of Spontaneously Broken Gauge Theories
  - Non-abelian SU(2) gauge theory ($\phi$ transforming in two different representations)
  - Patterns of symmetry breaking
  - EWSB (if time)
Symmetries of the Lagrangian - Global Symmetry

The Lagrangian of a system is given by:

\[ \mathcal{L} = \mathcal{L}(\phi_i(x), \partial_\mu \phi_i(x)) \]

If it is invariant under the transformation

\[ \phi_i(x) \to \phi_i'(x) = \phi_i(x) + \delta \phi_i(x) \]

where

\[ \delta \phi_i(x) = i \epsilon^a t^a_{ij} \phi_j(x) \]

where the \( \epsilon^a \) are spacetime independent small parameters and the \( t^a \) are a set of matrices satisfying the Lie algebra of the group

\[ [t^a, t^b] = i f^{abc} t^c \]

Then we say the transformation is a \textit{global} symmetry of the Lagrangian.
Symmetries of the Lagrangian - SSB

We use non-invariance of the ground state as a symmetry breaking condition.

Consider the linear sigma model: set of $N$ real scalar fields $\phi^i(x)$:

$$L = \frac{1}{2}(\partial_{\mu}\phi^i)^2 + \frac{1}{2}\mu^2(\phi^i)^2 - \frac{\lambda}{4}[(\phi^i)^2]^2$$

This Lagrangian is invariant under $O(N)$:

$$\phi^i \rightarrow R^{ij} \phi^j$$

The potential $V(\phi^i) = -\frac{1}{2}\mu^2(\phi^i)^2 + \frac{\lambda}{4}[(\phi^i)^2]^2$ with $\mu^2 > 0$ is minimized by any constant field $\phi^i_0$ which satisfies:

$$(\phi^i_0)^2 = \frac{\mu^2}{\lambda}$$
The constant field \( \phi^i_0 \) is the lowest-energy classical configuration of the system, with determined length but arbitrary direction.

Choose coordinates such that \( \phi^i_0 \) points in the \( N \)th direction:

\[
\phi^i_0 = (0, 0, \ldots, 0, v), \quad \text{where} \quad v = \frac{\mu}{\sqrt{\lambda}}
\]

Define a set of shifted fields, then rewrite the Lagrangian:

\[
\phi^i(x) = (\pi^k(x), v + \sigma(x)), \quad k = 1, \ldots, N - 1
\]

\[
\mathcal{L} = \frac{1}{2} (\partial_\mu \pi^k)^2 + \frac{1}{2} (\partial_\mu \sigma)^2 - \frac{1}{2} (2\mu^2) \sigma^2
\]

\[
- \sqrt{\lambda} \mu \sigma^3 - \sqrt{\lambda} \mu (\pi^k)^2 \sigma - \frac{\lambda}{4} \sigma^4 - \frac{\lambda}{2} (\pi^k)^2 \sigma^2 - \frac{\lambda}{4} [(\pi^k)^2]^2
\]
Symmetries of the Lagrangian - SSB

We obtain a massive $\sigma$ field, but the $N - 1$ $\pi$ fields remain massless and rotate among themselves.

The $O(N)$ symmetry is no longer manifest. Invariance of the ground state under $O(N)$ resulted in the spontaneous symmetry breaking of the Lagrangian. We have broken the $O(N)$ group down to $O(N - 1)$.

We can see this intuitively in the $O(2)$ case:
Symmetries of the Lagrangian - SSB

Generally, we see that although the potential is group invariant

\[ V(U(g)\phi) = V(\phi) \]

the point \( \phi_0 \), where \( V(\phi) \) takes it minimum is not group invariant

\[ U(g)\phi_0 \neq \phi_0 \quad \text{for some} \quad g \in G \]

And although \( \phi_0 \) is not invariant for all \( g \in G \), it may be invariant for some \( g \in G \). Those elements for which \( \phi_0 \) is invariant form a subgroup \( H \) of \( G \) and, in fact, \( H \) is the little group (or isotropy group or stability group) of \( \phi_0 \).

\[ U(h)\phi_0 = \phi_0 \iff h \in H \]

In the linear sigma model, we saw that \( G = O(N) \) and \( H = O(N-1) \).
Symmetries of the Lagrangian - Goldstone’s Theorem

Regarding the appearance of massless particles in the linear sigma model (the $\pi^k$):

*Goldstone’s Theorem*: Spontaneous breakdown of a continuous symmetry implies the existence of massless, spinless particles.

Every continuous symmetry of the Lagrangian that is not a symmetry of $\phi_0$ gives rise to a *Goldstone boson*, hence there were $N - 1$ massless particles in a linear sigma model with $N$ scalar fields.

See *Peskin & Schroeder, Cheng & Li*, etc. for proofs of Goldstone’s theorem.
Gauge Theories - Local symmetries

Now consider symmetries of the Lagrangian in which the parameters of the symmetry transformation are spacetime dependent:

\[ \epsilon^a \rightarrow \epsilon^a(x) \]

This is a local symmetry of the Lagrangian, also known as a gauge symmetry.

Gauging the symmetry complicates the transformation of the derivative term and requires the inclusion of a new vector field to keep the Lagrangian gauge invariant (that is, invariant under the local, spacetime dependent symmetry). This new vector field, called the gauge field, introduces dynamics into the theory.
Gauge Theories - Local symmetries

As an example, requiring gauge symmetry of the Dirac free electron theory introduces dynamics into the theory. Consider the Lagrangian for the free-electron field $\psi(x)$

$$\mathcal{L} = \bar{\psi}(x)(i\gamma^\mu \partial_\mu - m)\psi(x)$$

This has a global $U(1)$ symmetry:

$$\psi(x) \rightarrow \psi'(x) = e^{-i\alpha} \psi(x)$$

$$\bar{\psi}(x) \rightarrow \bar{\psi}'(x) = e^{i\alpha} \bar{\psi}(x)$$
Gauge Theories - Local symmetries

Now gauge the symmetry by replacing $\alpha$ with $\alpha(x)$, so the transformations become:

$$ \psi(x) \rightarrow \psi'(x) = e^{-i\alpha(x)}\psi(x) $$

$$ \bar{\psi}(x) \rightarrow \bar{\psi}'(x) = e^{i\alpha(x)}\bar{\psi}(x) $$

The derivative term now has a complicated transformation that is not locally invariant:

$$ \bar{\psi}(x)\partial_\mu \psi(x) \rightarrow \bar{\psi}'(x)\partial_\mu \psi'(x) = \bar{\psi}(x)e^{i\alpha(x)}\partial_\mu (e^{-i\alpha(x)}\psi(x)) $$

$$ = \bar{\psi}(x)\partial_\mu \psi(x) - i\bar{\psi}(x)\partial_\mu \alpha(x)\psi(x) $$

$$ = \bar{\psi}(x)(\partial_\mu \psi(x) - i\partial_\mu \alpha(x))\psi(x) $$
Gauge Theories - Local symmetries

Form a gauge-covariant derivative $D_\mu$ to replace $\partial_\mu$, such that $D_\mu \psi(x)$ transforms like

$$D_\mu \psi(x) \rightarrow [D_\mu \psi(x)]' = e^{-i\alpha(x)} D_\mu \psi(x)$$

Accomplish this by introducing the gauge field $A_\mu(x)$ and define the covariant derivative

$$D_\mu \psi(x) = (\partial_\mu + ieA_\mu) \psi(x)$$

where $A_\mu$ transforms like

$$A_\mu(x) \rightarrow A'_\mu(x) = A_\mu(x) + \frac{1}{e} \partial_\mu \alpha(x)$$
Gauge Theories - Local symmetries

We now have a Lagrangian that is invariant under a local $U(1)$ symmetry (gauge invariance!)

$$\mathcal{L}_0 = \bar{\psi}(i\gamma^\mu \partial_\mu + ieA_\mu)\psi - m\bar{\psi}\psi$$

To complete the theory, make the gauge field dynamical by adding a term to the Lagrangian containing its derivatives:

$$\mathcal{L}_A = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu}, \quad \text{where} \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

We can see this term is invariant using the transformation property of $A_\mu$ or from the relation of $F_{\mu\nu}$ to the covariant derivative:

$$[D_\mu, D_\nu]\psi = ieF_{\mu\nu}\psi$$
Combining $\mathcal{L}_0$ and $\mathcal{L}_A$, we get the full QED Lagrangian:

$$\mathcal{L}_{QED} = \overline{\psi}(i\gamma^\mu \partial_\mu + ieA_\mu)\psi - m\overline{\psi}\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}$$

The dynamics of the theory come from requiring gauge invariance of the Lagrangian, that is, by requiring the Dirac free electron theory to be invariant to \textit{local} $U(1)$ transformations.

Note the following features of the theory:

1. The photon is massless because an $A_\mu A^\mu$ term is not gauge invariant.
2. The coupling of the photon to any matter field is determined by its transformation property under the symmetry group.
3. The QED Lagrangian does not have gauge field self-coupling.
Now let’s combine local gauge invariance and spontaneous symmetry breaking: The Higgs Mechanism

Consider an Abelian $U(1)$ gauge theory with a complex scalar field coupled to itself and to an electromagnetic field:

$$\mathcal{L} = -\frac{1}{4}(F_{\mu\nu})^2 + |D_\mu \phi|^2 + \mu^2 \phi^* \phi - \frac{\lambda}{2} (\phi^* \phi)^2$$

With $\mu^2 > 0$, the minimum of the potential occurs at:

$$v = \langle \phi \rangle = \phi_0 = \sqrt{\frac{\mu^2}{\lambda}}$$

where $\langle \phi \rangle = |\langle 0 | \phi | 0 \rangle|$ is the vacuum expectation value (vev) of the field $\phi$. 
Gauge Theories - SSB

We can write $\phi$ in terms of the real fields $\phi_1$ and $\phi_2$

$$\phi = \frac{1}{\sqrt{2}}(\phi_1 + i\phi_2)$$

choose

$$\langle 0 | \phi_1 | 0 \rangle = v \quad \text{and} \quad \langle 0 | \phi_2 | 0 \rangle = 0$$

and define the shifted fields

$$\phi'_1 = \phi_1 - v \quad \text{and} \quad \phi'_2 = \phi_2$$

All this follows the procedure followed earlier for SSB of a global symmetry. So far we have one massive field $\phi'_1$ (analogous to $\sigma$) while $\phi'_2$ corresponds to the massless Goldstone boson. However...
The local gauge symmetry adds a new feature. The covariant derivative term will yield

\[
|D_\mu \phi|^2 = |(\partial_\mu - igA_\mu)\phi|^2
= \frac{1}{2}(\partial_\mu \phi'_1 + gA_\mu \phi'_2)^2 + \frac{1}{2}(\partial_\mu \phi'_2 - gA_\mu \phi'_1)^2
- gvA_\mu (\partial_\mu \phi'_2 + gA_\mu \phi'_1) + \frac{g^2v^2}{2}A^\mu A_\mu
\]

At this point, it appears the gauge boson acquires a mass \(m_A = gv\). But the interaction term \(gvA_\mu \partial_\mu \phi'_2\) makes the picture a little unclear.
Gauge Theories - SSB

We now make a particular choice of gauge, called the unitary gauge to eliminate the massless Goldstone boson from the theory. Use the local $U(1)$ gauge symmetry to choose $\alpha(x)$ so that $\phi(x)$ is real-valued at every point $x$ (see Cheng & Li, p242). With this choice, $\phi'_2$ is removed from the theory, leaving the Lagrangian as:

$$\mathcal{L} = -\frac{1}{4} (F_{\mu\nu})^2 + (\partial_\mu \phi)^2 + g^2 \phi^2 A_\mu A^\mu - V(\phi)$$

Before the symmetry breaking, we had two scalar fields and one massless gauge boson, so a total of four degrees of freedom. After the symmetry breaking, we have one scalar field, and one massive gauge boson, still a total of four degrees of freedom.
Non-Abelian Gauge Theories

We can generalize gauge theories and the Higgs mechanism to non-Abelian gauge symmetries. Consider an $SU(2)$ gauge theory with a complex doublet of scalar fields $\phi = (\phi_1 \phi_2)$ which transform as a spinor of $SU(2)$

$$\mathcal{L} = \frac{1}{4} (F_{\mu\nu})^2 + |D_\mu \phi|^2 - V(\phi)$$

where

$$D_\mu \phi = \left( \partial_\mu - ig \frac{\sigma^a}{2} A^a_\mu \right) \phi$$

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g \epsilon^{abc} A^b_\mu A^c_\nu$$

$$V(\phi) = -\mu^2 \phi^* \phi + \frac{\lambda}{2} (\phi^* \phi)^2$$
Non-Abelian Gauge Theories

For $\mu^2 > 0$, the potential acquires a vev, and using the freedom of $SU(2)$ rotations, we can write this as:

$$\langle \phi \rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v \end{pmatrix}$$

The gauge boson masses arise from

$$|D_\mu \phi|^2 = \frac{1}{2} g^2 (0 \ v) \frac{\sigma^a}{2} \frac{\sigma^b}{2} \begin{pmatrix} 0 \\ v \end{pmatrix} A^a_\mu A^{b\mu} + \ldots$$

Symmetrizing the matrix product using $\{\sigma^a, \sigma^b\} = 2\delta^{ab}$, we find the mass term

$$\Delta \mathcal{L} = \frac{g^2 v^2}{8} A^a_\mu A^{a\mu}$$
Non-Abelian Gauge Theories

In the case of the spinor rep. in $SU(2)$ all three gauge bosons receive the mass $m_A = \frac{gv}{2}$.

If instead we take $\phi$ to be real-valued under $SU(2)$ and transform according to the vector rep, the appropriate covariant derivative is

$$(D_\mu \phi)_a = \partial_\mu \phi_a + g\epsilon_{abc} A^b_\mu \phi_c$$

Again, when $\phi$ acquires a vev, we square the $D_\mu \phi$ term and find the gauge boson mass term

$$\Delta \mathcal{L} = \frac{g^2}{2} (\epsilon_{abc} A^b_\mu (\phi_0)_c)^2$$
Non-Abelian Gauge Theories

Using the freedom in $SU(2)$ to pick a direction for the vev, choose it to point in the 3 direction

$$\langle \phi_c \rangle = (\phi_0)_c = V \delta_{c3}$$

Inserting this into $\Delta \mathcal{L}$

$$\Delta \mathcal{L} = \frac{g^2}{2} V^2 (\epsilon_{abc} A^b_\mu)^2 = \frac{g^2}{2} V^2 \left( (A^1_\mu)^2 + (A^2_\mu)^2 \right)$$

In the case of the vector rep. in $SU(2)$ the gauge bosons corresponding to generators 1 and 2 acquire masses, while the boson corresponding to generator 3 remains massless

$$m_1 = m_2 = gV, \quad m_3 = 0$$

With $\langle \phi_c \rangle$ pointing along the 3-axis, it is still invariant under $SO(2) \cong U(1)$, hence two bosons acquire mass.
Pattern of Symmetry Breaking

The pattern of symmetry breaking depends on the structure of the theory, particularly the group representation of the scalar field (see, for instance, Li:)

<table>
<thead>
<tr>
<th>Representation</th>
<th>$O(n)$</th>
<th>$SU(n)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>vector</td>
<td>$O(n)$</td>
<td>$SU(n-1)$</td>
</tr>
<tr>
<td>$k$ vectors</td>
<td>$O(n-k)$</td>
<td>$SU(n-k)$</td>
</tr>
<tr>
<td>2nd rank symmetry tensor</td>
<td>$O(n-1)$ or $O(k) \times O(n-k)$</td>
<td>$SU(n-1)$ or $O(n)$</td>
</tr>
<tr>
<td>2nd rank anti-symmetric tensor</td>
<td>$U(k)$ or $U(1) \times O(n-2)$</td>
<td>$O(2k+1)$ or $SU(n-2)$</td>
</tr>
<tr>
<td>adjoint representation</td>
<td>$SU(1) \times SU(n-k) \times U(1)$</td>
<td>$SU(n-1)$</td>
</tr>
</tbody>
</table>

TABLE 3

Summary of the pattern of symmetry breaking
Summary

• Physical Lagrangians have internal symmetries (they are invariant under certain group transformations)

• Invariance of the ground state is a condition for spontaneously breaking symmetries of $\mathcal{L}$

• A Gauge theory is one in which we require $\mathcal{L}$ to be invariant under a local symmetry, and this results in one gauge boson per generator of the local symmetry group

• Spontaneous symmetry breaking of a gauge theory (The Higgs Mechanism) involves:
  1. SSB a global symmetry of the system (the symmetry that the scalar field $\phi$ transforms under)
  2. Choose a gauge to remove Goldstone bosons (and timelike components of gauge bosons) from the theory

• We can categorize spontaneous symmetry breaking for various groups (and various representations of the groups)
References


