Magnetic Monopoles and the Homotopy Groups

Physics 251
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Solitons

- Finite-energy, non-dissipative solutions of the classical wave equations
- Cannot arise in linear theories such as electrodynamics, because there is dispersion
- Non-linear theories allow cancellation of dispersive effects
M is the set of vacuum field configurations:

\[ E(\Phi) - E(\Phi_0) < +\infty \quad (\Phi_0 \in M) \quad \text{(finite energy)} \]

\[ \partial_t \Phi = 0 \quad \text{(static)} \]

\[ \lim_{r \to \infty} \Phi_a(r\vec{n}) = \lambda_a(\vec{n}) \in M \quad \text{(asymptotic vacuum value)} \]

Define a topological quantum number \( \pi_{d-2}(G/H) \), where \( G \) maps one vacuum to another and \( H \) is the isotopy group (\( G/H \) is the manifold of the vacuum)
Solitons in 1+1 Dimensions

\[ \mathcal{L} = \frac{1}{2} (\partial_\mu \phi)^2 - V(\phi) \quad V(\phi) = -\frac{1}{2} m^2 \phi^2 + \lambda \phi^4 \]

Euler—Lagrange Equations give us the equation of motion

\[ \frac{\partial^2 \phi}{\partial t^2} - \frac{\partial^2 \phi}{\partial x^2} = m^2 \phi - \lambda \phi^3 \]

This has static solutions at zero and the potential minimum

\[ \phi = 0, \pm \frac{\mu}{\sqrt{\lambda}} \]
Kinky Soliton Solutions

- Equation of motion can also be satisfied by the soliton kink solution:

\[ \varphi_{\text{kink}}(x) = \varphi(x - x_0) = \pm \frac{\mu}{\sqrt{\lambda}} \tanh \left( \frac{\mu}{\sqrt{2}} (x - x_0) \right) \]
Topological Charge

- The scalar fields in 2D have a conserved current, leading to a topological charge:

\[ b_\mu^* = \epsilon_{\mu \nu} b^\nu, \quad b_\nu = \partial_\nu \varphi \quad \frac{d}{dt} \int_{-\infty}^{\infty} dx b_\mu^* = 0 \quad \text{or} \quad Q \equiv \varphi(+\infty, t) - \varphi(-\infty, t) \]

- For the kink solution:

<table>
<thead>
<tr>
<th>Charge</th>
<th>Topological Quantum Number</th>
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<tbody>
<tr>
<td>[ Q = \frac{2\mu}{\sqrt{\lambda}} ]</td>
<td>[ \pi_0 \left( \frac{Z_2}{e} \right) = Z_2 ]</td>
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Dirac Monopoles

To construct a magnetic monopole, you can have $\mathbf{A}$ of the form

$$
\vec{A}_N = \frac{g}{4\pi r} \frac{1 - \cos(\theta)}{\sin(\theta)} \hat{\phi} \\
\vec{A}_S = \frac{g}{4\pi r} \frac{1 + \cos(\theta)}{\sin(\theta)} \hat{\phi}
$$

$$
\vec{B} = \nabla \times \vec{A} = \frac{g}{4\pi r^2} \hat{r}
$$

Magnetic Monopole with charge $g$
Singularities at $\theta=0$ and $\theta=\pi$ correspond to Dirac string.

Moving in a circle around string, particle wave function picks up a phase ($e^{-ieg}$).

This phase factor must be equal to 1 for string to be undetectable:

$$e^{-ieg} = 1 \quad \text{and} \quad eg = 2\pi n$$
From previous calculation we see that charge is quantized

Monopoles require a compact U(1) gauge group

Can be embedded in a larger compact gauge group i.e. SU(2)

Also can arise in Kaluza-Klein theories
Soliton in 2+1 Dimensions (Nielsen-Olsen Model)

\[ L = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} |D_\mu \phi|^2 + \frac{1}{2} \mu^2 |\phi|^2 - \frac{1}{4} \lambda |\phi|^4 \]

- Here \( D_\mu = \partial_\mu - ieA_\mu \) and \( \Phi \) is a complex valued field
- Vacuum is identified with \( F_{\mu\nu} = 0, D_\mu \phi = 0, |\phi| = \frac{\mu}{\sqrt{\lambda}} \)
- The vacuum fields have the values \( \mu \sqrt{\lambda} e^{i\chi(\theta)} \)

\[ \pi_{d-2}(G/H) = \pi_1(S^1) = \mathbb{Z} \]
Boundary Conditions

- As \( r \to \infty \), we need to have vacuum states. Working in gauge where \( A^0 = 0 \) and considering the requirements on vacuum solution:
  \[
  A_r \to 0, \quad \frac{1}{r} \frac{d \chi}{d \theta} \to e A_\theta, \quad \text{or} \quad A_\theta \to \frac{1}{e r} \frac{d \chi}{d \theta}
  \]

- Defining \( \chi(\theta) = n \theta \) where \( n \in \mathbb{Z} \) refers to a homotopy class we find that magnetic field through (x,y) plane is quantized.

\[
\Phi = \int H(x, y, t) \, dx \, dy = \int_{S^1} \vec{A} \cdot d\vec{l} = \frac{1}{e} \int_0^{2\pi} \frac{1}{r} \frac{d \chi}{d \theta} \, r d\theta = \frac{2\pi n}{e}
\]
\[ L = -\frac{1}{4} G^{a}_{\mu \nu} G^{a\mu \nu} + \frac{1}{2} (D_{\mu} \varphi^{a}) (D^{\mu} \varphi^{a}) + \frac{1}{2} (\varphi^{a} \varphi^{a}) - \frac{1}{4} \lambda (\varphi^{a} \varphi^{a})^{2} \]

- \( \Phi^{a} \) are scalar fields in the adjoint representation of SU(2) (a=1,2,3).

\[
G^{a}_{\mu \nu} = \partial_{\mu} \varphi^{a} - \partial_{\nu} \varphi^{a} + e \varepsilon_{abc} W_{\mu b} W_{\nu c}
\]

\[
D_{\mu} \varphi^{a} = \partial_{\mu} \varphi^{a} + e \varepsilon^{abc} W_{\mu b} \varphi^{c}
\]

- The vacuum state manifold is a 2 sphere with radius \( \phi \cdot \phi = \varphi^{a} \varphi^{a} = \mu / \sqrt{\lambda} \)

- \( \pi_{2}(SO(3) / SO(2)) = \pi_{1}(SO(2)) = \mathbb{Z} \)
Boundary Conditions

- $r \to \infty$, need vacuum state, $D_\mu \phi^a = 0$
  $$W_\mu^a(\vec{x}, t) \to -\epsilon_{\mu ab} \frac{r_b}{er^2} \varphi^a(\vec{x}, t) \to \frac{r_a}{r} \mu / \sqrt{\lambda}$$

- Shown by 't Hooft that $W_\mu^a$ and $F_{\mu \nu}$ are related by ($\Phi = \{\phi^a\}$):
  $$F_{\mu \nu} = \phi^a G_{\mu \nu}^a - \frac{1}{e} \epsilon_{abc} \varphi^a (D_\mu \phi^b) (D_\nu \phi^c) / |\Phi|^3$$

- Working out electric and magnetic fields using the $r \to \infty$ limit value of $W_\mu^a$
  $$E_i = -F_{0i} = 0$$
  $$H_i = -\frac{1}{2} \epsilon_{ijk} F_{jk} = \frac{1}{er^3} r_i$$
Quantization of Magnetic Charge

\[ \int \vec{H} \cdot d\vec{s} = +\frac{1}{e} \int \frac{1}{r^2} r^2 d\Omega = \frac{4\pi}{e} = 4\pi g \]

- This is same as Dirac quantization condition (\(eg = 2\pi n\)) with \(n = 2!\)
- Considering all possible solutions of the equations of motion for this theory, it can be shown we get the Dirac quantization condition with \(n\) even
- Agrees with result for possible homotopy classes
Mass of Monopoles

- t’ Hooft showed \( M_{mon} \approx \frac{4\pi M_V}{e^2} \), where \( M_V \) is gauge boson mass.

- Monopoles arise in GUTs, with mass of the scale of the symmetry breaking of the theory.

- \( SU(5) \xrightarrow{M_X} SU(3) \times SU(2) \times U(1) \xrightarrow{M_W} SU(3) \times U(1) \)

- GUT monopoles are too massive to be produced in accelerators, but could have been produced in early universe.

- Monopole searches focus on accelerators, cosmic rays, and on monopoles possibly bound in matter.

- To date, nothing found.
References