The Quark Model and SU(3)

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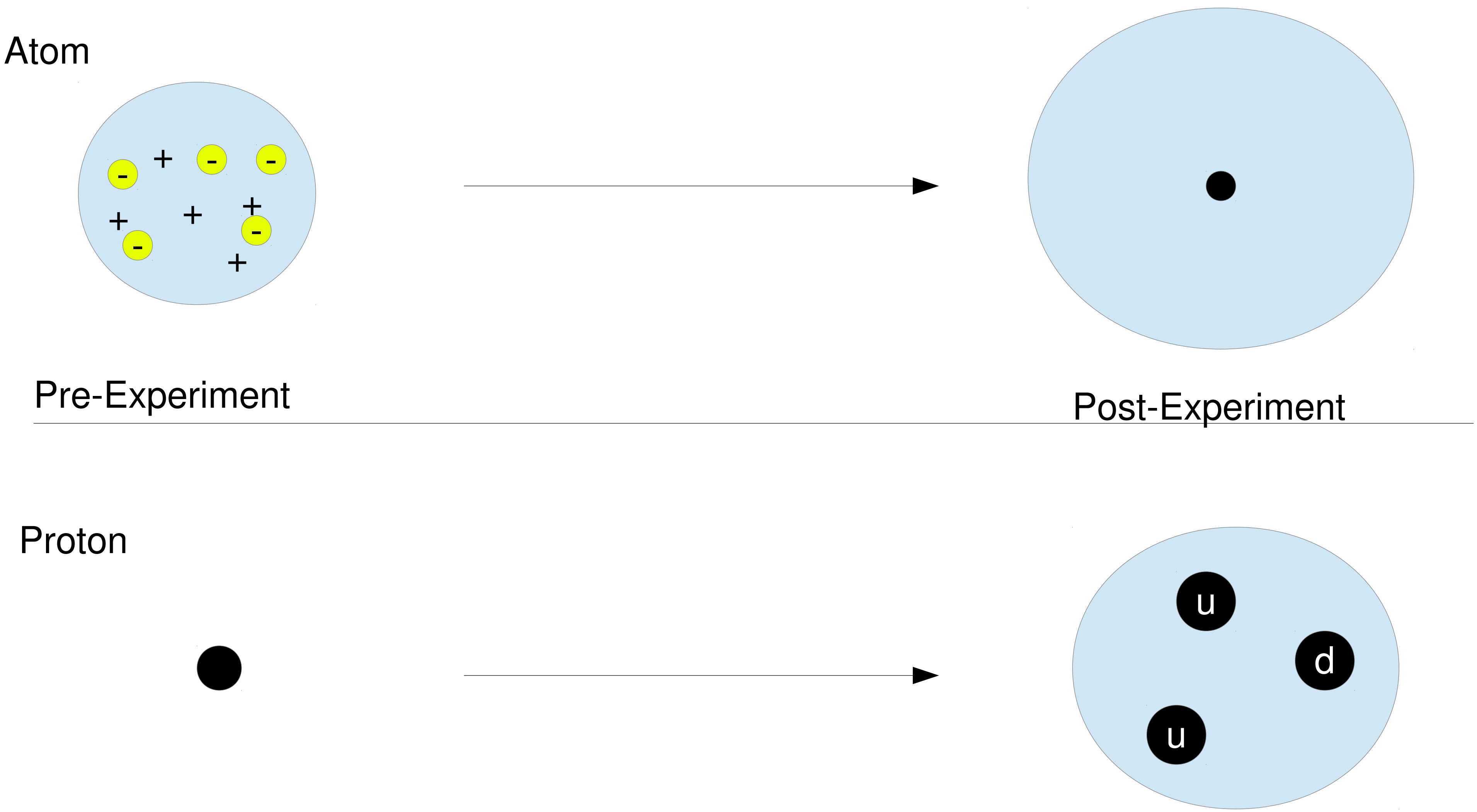
Physics 251

Scattering Experiments at SLAC in the 1960s suggested that pointlike particles exist in the proton

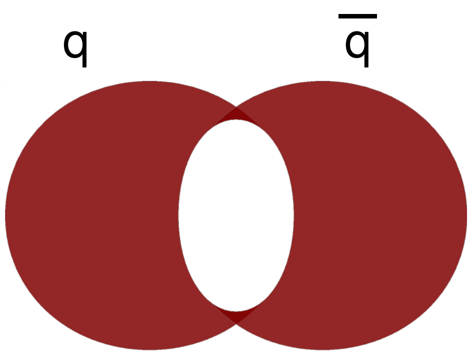


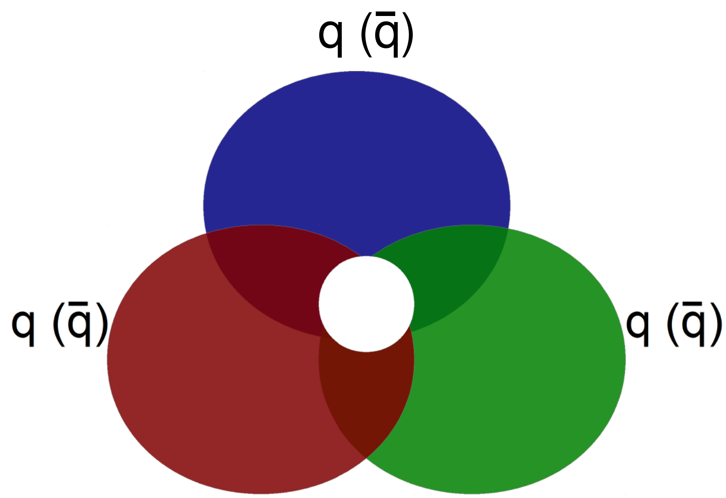
Six quarks and six anti-quarks have been discovered

	d	u	s	c	b	t
Q – electric charge	$-\frac{1}{3}$	$+\frac{2}{3}$	$-\frac{1}{3}$	$+\frac{2}{3}$	$-\frac{1}{3}$	$+\frac{2}{3}$
I – isospin	$\frac{1}{2}$	$\frac{1}{2}$	0	0	0	0
I_z – isospin z-component	$-\frac{1}{2}$	$+\frac{1}{2}$	0	0	0	0
S – strangeness	0	0	−1	0	0	0
C – charm	0	0	0	+1	0	0
B – bottomness	0	0	0	0	−1	0
T – topness	0	0	0	0	0	+1

J. Beringer *et al.*(PDG), PR **D86**, 010001 (2012) (http://pdg.lbl.gov)

An additional quantum number – color – restricts combinations of quarks. In nature, we observe $q\bar{q}$ and qqq





The free quark Lagrangian density has a SU(n) symmetry for n quarks*

$$L = \bar{q}^{a} i \gamma^{\mu} \partial_{\mu} q_{a} - \bar{q}^{a} m q_{a}$$

* with identical masses

First, let's consider a model with only up and down quarks.

In this case, the relevant quantum number is isospin with an SU(2) symmetry.

Generators of the 2-dimensional representation of SU(2):

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad , \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad , \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Commutation Relations:

$$\left[\frac{1}{2}\sigma_i, \frac{1}{2}\sigma_j\right] = i\epsilon_{ijk}\left(\frac{1}{2}\sigma_k\right)$$

Raising/Lowering Operators

$$\sigma_\pm = \frac{1}{2}\left(\sigma_1 \pm i\sigma_2\right)$$

There are two eigenstates of $\frac{1}{2}\sigma_3$

$$\lambda_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad , \quad \lambda_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\frac{1}{2}\sigma_3\lambda_1 = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{2}\lambda_1 \quad , \quad \frac{1}{2}\sigma_3\lambda_2 = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{-1}{2}\lambda_2$$

In the 2 representation, u corresponds to λ_1 and d corresponds to λ_2

In the 2* representation,-$\bar{d}$ corresponds to λ_1 and $\bar{u}$ corresponds to λ_2

Meson states are formed from $2 \otimes 2^*$

$2 \otimes 2^* = 3 \oplus 1 \rightarrow$ We expect 4 meson states

$$Let \quad \psi = \begin{pmatrix} u \\ d \end{pmatrix} \quad , \quad \bar{\psi} = \begin{pmatrix} -\bar{d} \\ \bar{u} \end{pmatrix}$$

$$Single\ State: s = \bar{\psi}_\alpha \psi^\alpha$$

$$s \propto (u\bar{u} + d\bar{d})$$

$$Triplet\ State: v_k = \bar{\psi}_i (\sigma^k)^i{}_j \psi^j$$

$$v_1 \propto (\bar{u}d + \bar{d}u)$$

$$v_2 \propto (\bar{u}d - \bar{d}u)$$

$$v_3 \propto (\bar{d}d - \bar{u}u)$$

With appropriate linear combinations and normalization, we recover the pions ...

$$\pi^+ = u\bar{d} \quad , \quad \pi^- = \bar{u}d \quad , \quad \pi^0 = \frac{1}{\sqrt{2}}(u\bar{u} - d\bar{d})$$

...which fit into a 3-dim representation of SU(2)

$$\pi^+ = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad , \quad \pi^- = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \quad , \quad \pi^0 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

$$S_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \quad , \quad S_2 = \frac{i}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \quad , \quad S_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

In the quark model, we consider three quarks – up, down, and strange

$$m_u = 2.3^{+0.7}_{-0.5} \text{ MeV}$$

$$m_d = 4.8^{+0.7}_{-0.3} \text{ MeV}$$

$$m_s = 95 \pm 5 \text{ MeV}$$

$$m_c = 1.275 \pm 0.025 \text{ GeV}$$

$$m_b = 4.68 \pm 0.03 \text{ GeV}$$

$$m_t = 173.5 \pm 0.6 \pm 0.8 \text{ GeV}$$

There are 8 matrices in the defining representation of SU(3)

$$\phi = \begin{pmatrix} u \\ d \\ s \end{pmatrix} \quad , \quad \phi' = U\phi \quad , \quad U = \exp\left(\frac{1}{2} i \theta \hat{\boldsymbol{n}} . \boldsymbol{\lambda}\right)$$

$$\lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} , \quad \lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} , \quad \lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} , \quad \lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

$$\lambda_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix} , \quad \lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} , \quad \lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix} , \quad \lambda_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

Two of these matrices are diagonal

$$\left(\frac{1}{2}\right)\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \rightarrow \quad isospin: I_3$$

$$\left(\frac{1}{3}\right)\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix} \rightarrow \quad hypercharge: Y$$

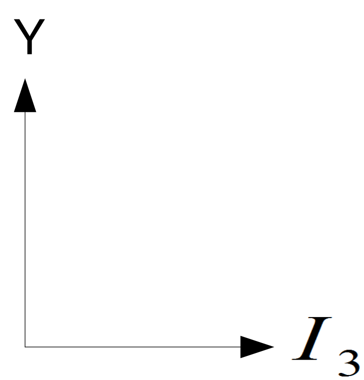
$$Q = I_3 + \frac{1}{2} Y$$

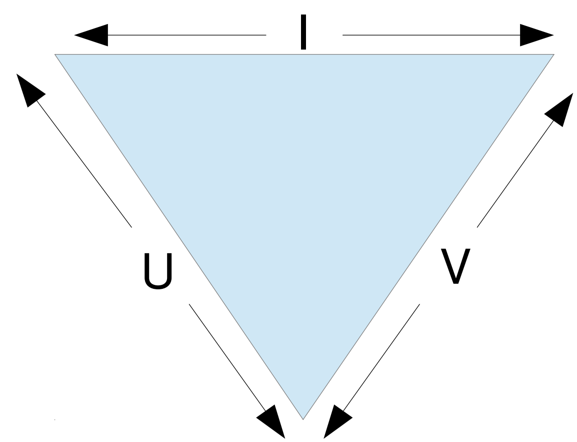
SU(3) contains pieces of SU(2)

$$\lambda_{1,2} = \begin{pmatrix} \sigma_{1,2} & 0 \\ 0 & 0 \end{pmatrix} \rightarrow \; isospin$$

$$\lambda_{6,7} = \begin{pmatrix} 0 & 0 \\ 0 & \sigma_{1,2} \end{pmatrix} \rightarrow \; U-spin$$

$$\lambda_{4,5} \qquad \rightarrow \; V-spin$$





We can form raising and lowering operators within each subgroup

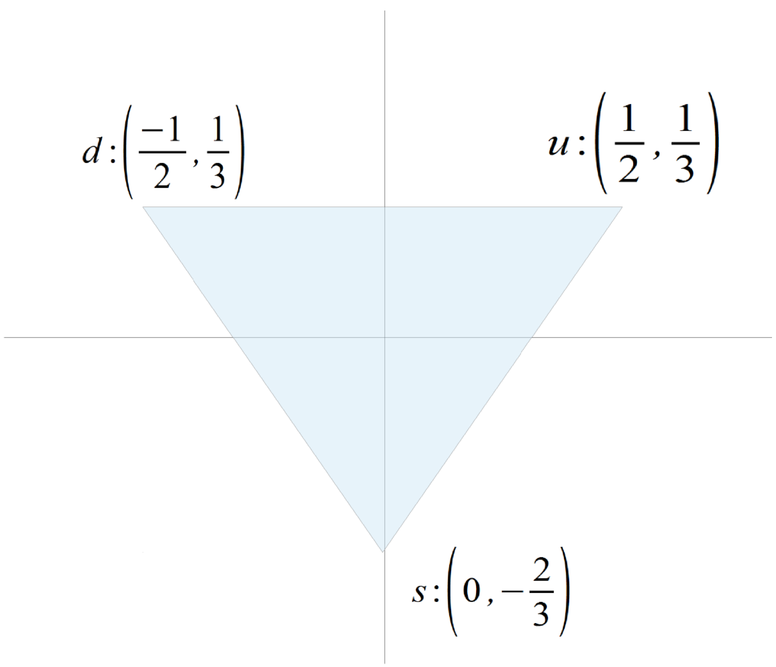
$$I_{\pm} = \frac{1}{2}\left(\lambda_1 \pm i\lambda_2\right) \quad , \quad u \leftarrow\rightarrow d$$

$$v_{\pm} = \frac{1}{2}\left(\lambda_4 \pm i\lambda_5\right) \quad , \quad s \leftarrow\rightarrow u$$

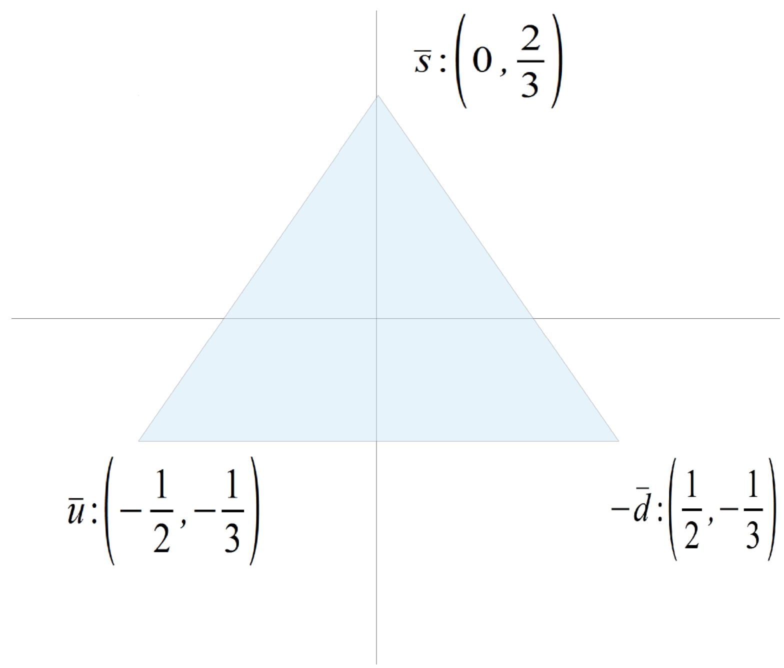
$$u_{\pm} = \frac{1}{2}\left(\lambda_6 \pm i\lambda_7\right) \quad , \quad s \leftarrow\rightarrow d$$

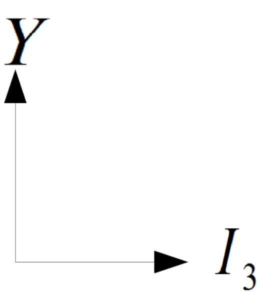
Using the eigenvalues of I_3 and Y, we can draw weight diagrams

3

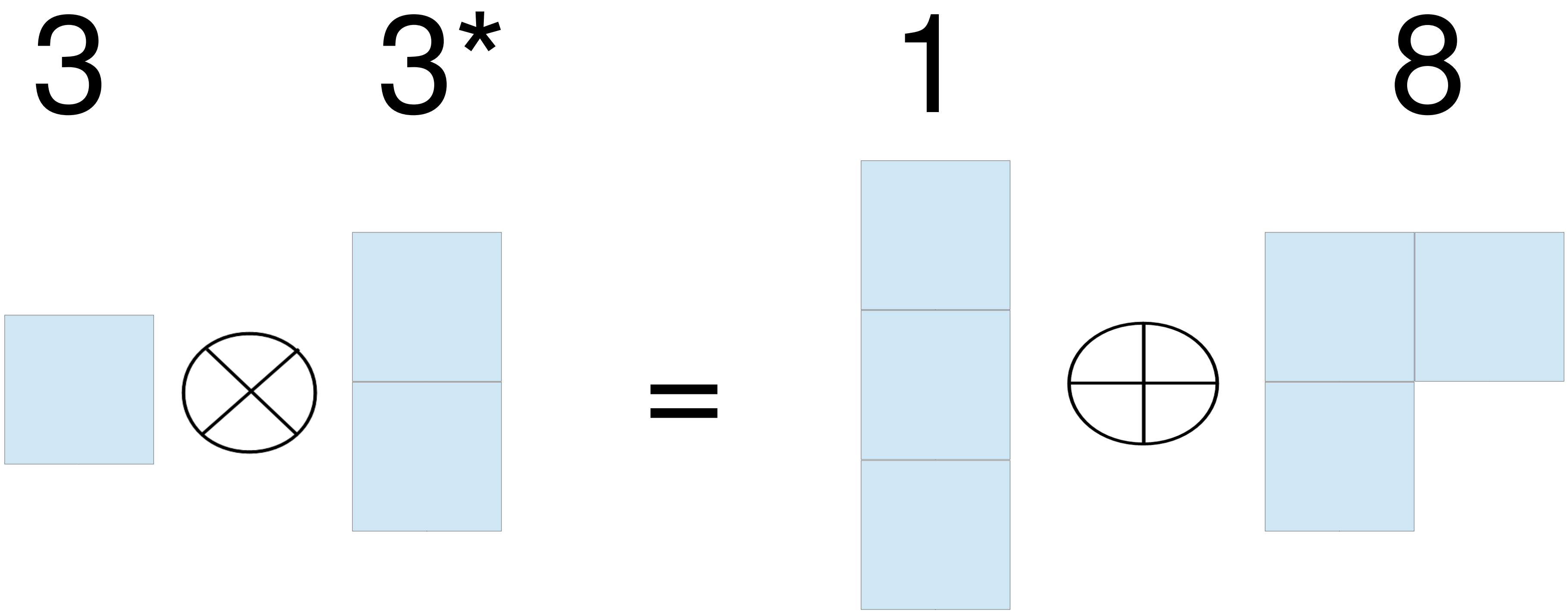


3*



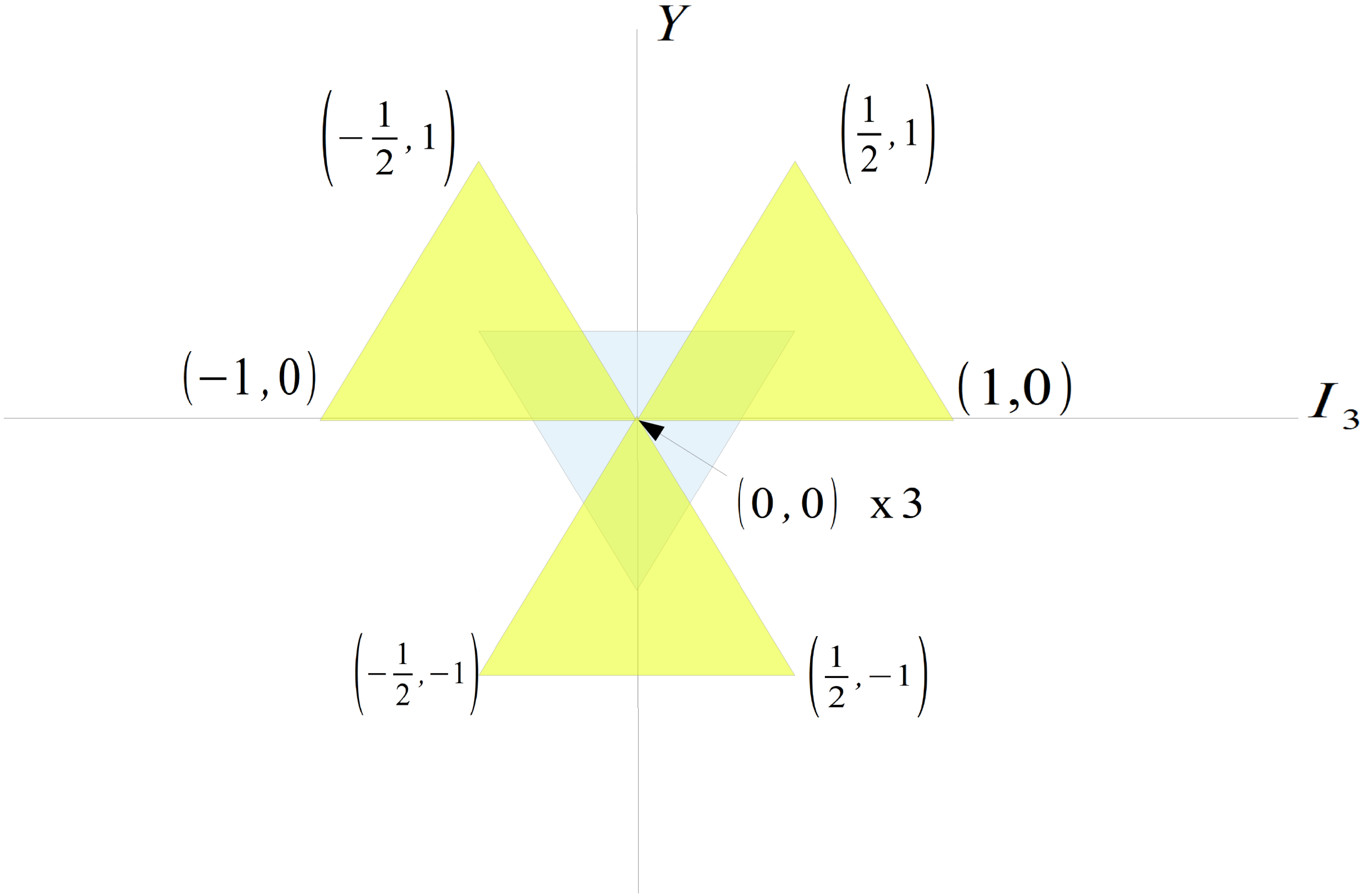


Meson states are $3 \otimes 3^*$

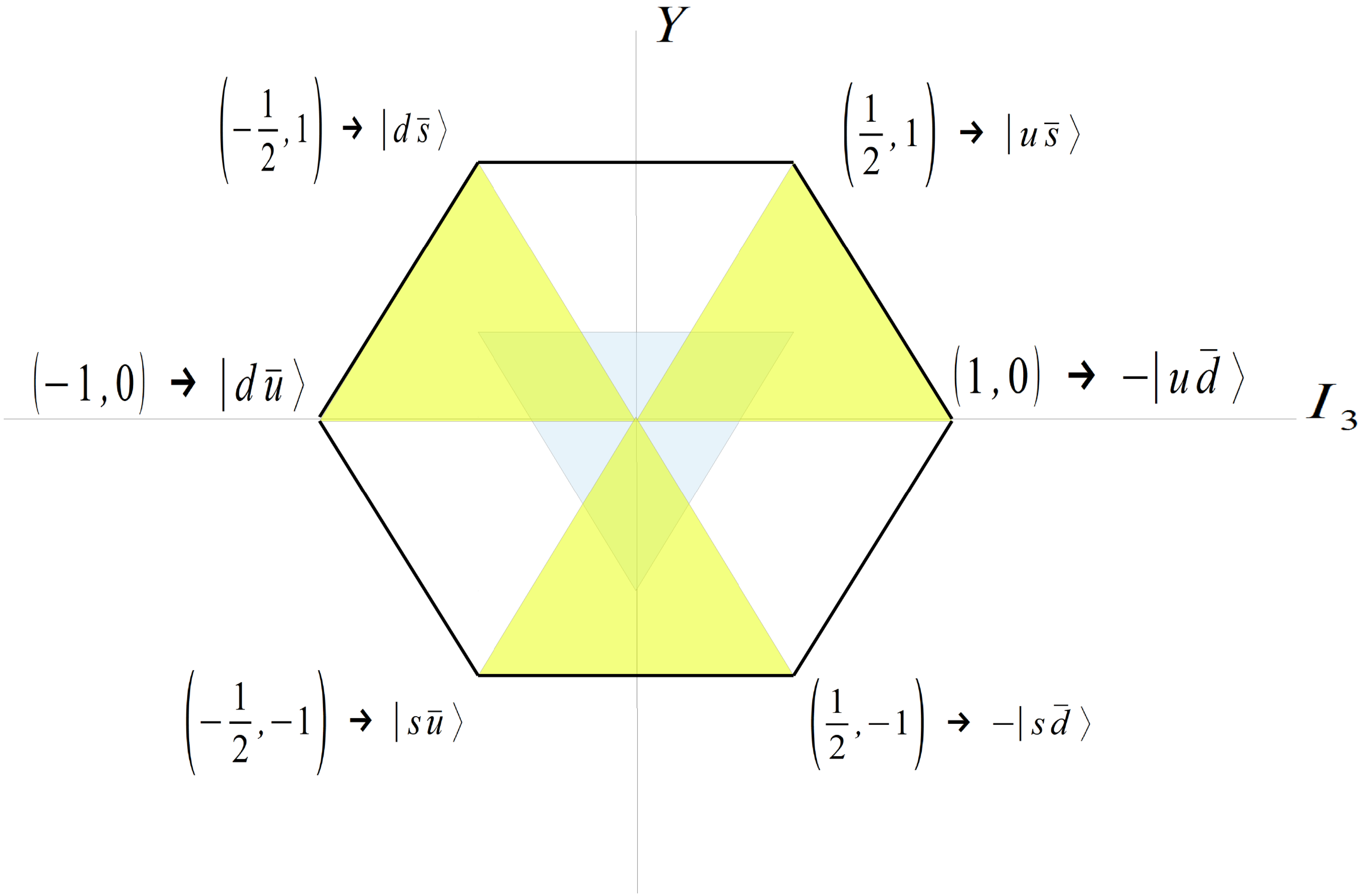


→ We expect 9 states!

To find states, overlay 3* on 3 in the weight diagram



States on the hexagon vertices have a unique quark content



One (0,0) state is found using raising/lowering operators

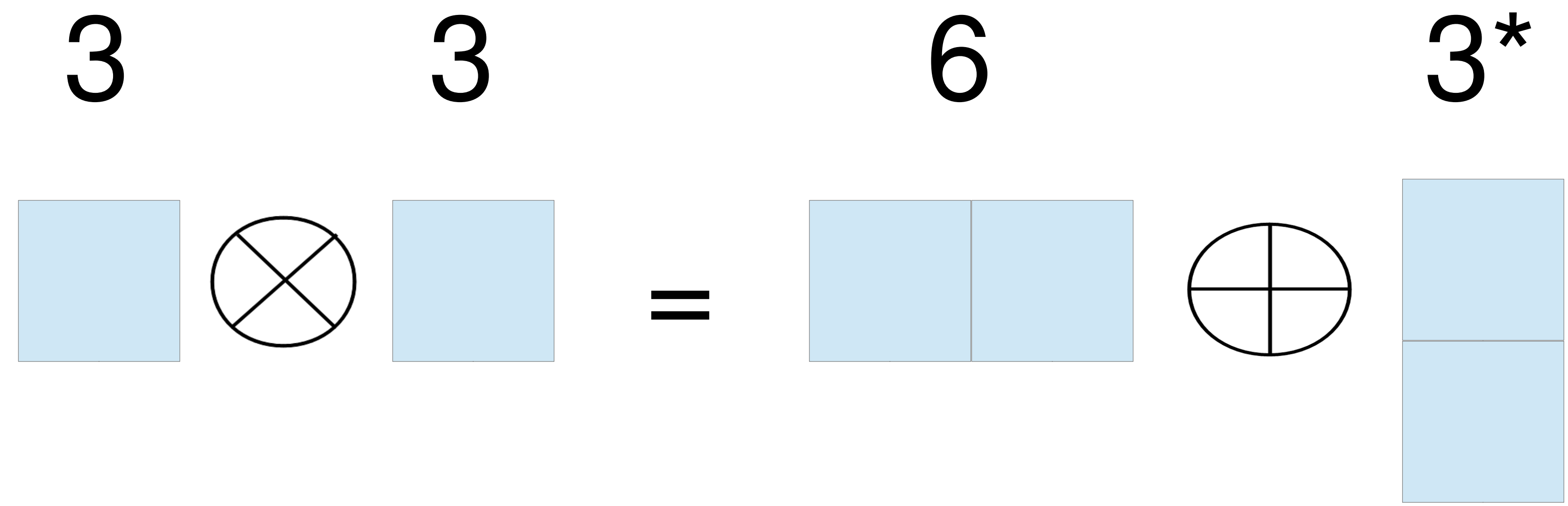
$$\frac{1}{\sqrt{2}} I_{-}\left(-|u\bar{d}\rangle\right) = \frac{1}{\sqrt{2}}\left(|u\bar{u}\rangle - |d\bar{d}\rangle\right)$$

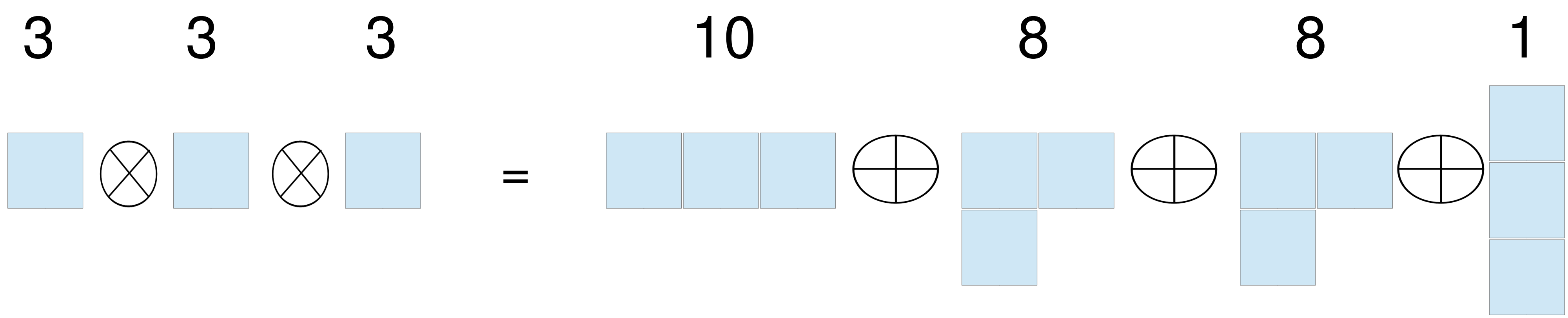
Other (0,0) states are orthogonal

$$C_1\left(|u\bar{u}\rangle + |d\bar{d}\rangle + C_2|s\bar{s}\rangle\right)$$

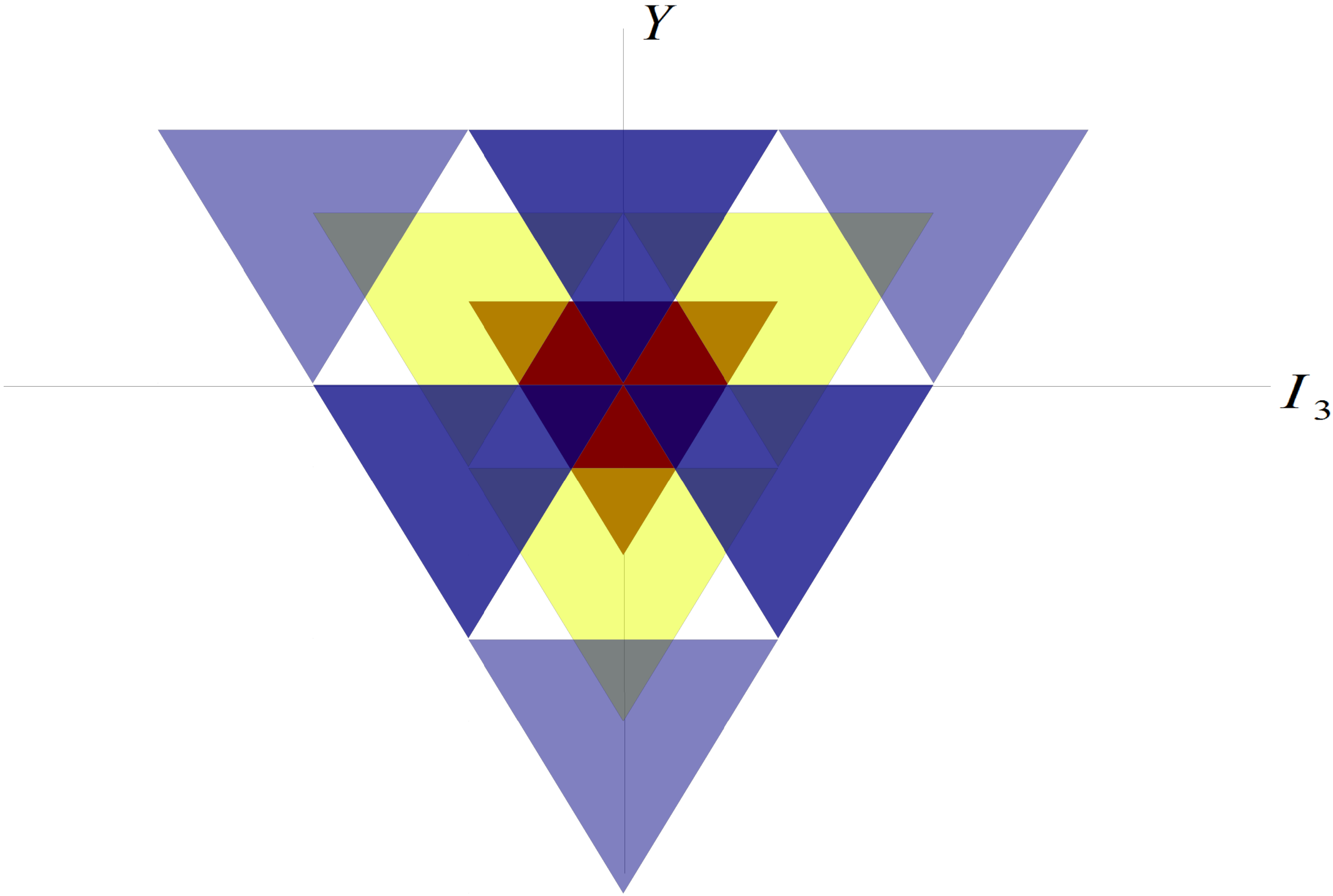
$$C_3\left(|u\bar{u}\rangle + |d\bar{d}\rangle + C_4|s\bar{s}\rangle\right)$$

Baryon states are $3 \otimes 3 \otimes 3$





In this case, the overlaid weight diagram is more difficult



Using our knowledge of SU(3), we construct each N-let

Let $S_{jk} = \frac{1}{\sqrt{2}}\left(q_j q_k + q_k q_j\right)$, $A_{jk} = \frac{1}{\sqrt{2}}\left(q_j q_k - q_k q_j\right)$

Decouplet: $T_{ijk} = \frac{1}{\sqrt{3}}\left(S_{ij} q_k + S_{jk} q_i + S_{ki} q_j\right)$

Octet: $B^i_j = \frac{1}{2}\left[\epsilon^{ilm} A_{lm} q_j - \frac{\sqrt{2}}{3}\delta^i_j \epsilon^{klm} A_{lm} q_k\right]$

Octet': $B'^i_j = \frac{1}{\sqrt{3}}\epsilon^{ikl} S_{jl} q_k$

Singlet: $\frac{1}{2\sqrt{3}}\epsilon^{klm} A_{lm} q_k$

In practice, it is helpful to consider other physical properties

- Combining three spin-1/2 particles can only result in spin-3/2 or spin-1/2
- Particles of half integer spin must have completely anti-symmetric wave functions
- The color state is anti-symmetric

$\rightarrow$ The combination of flavor and spin must be symmetric

For a spin-up proton:

$$
\begin{aligned}
|p\uparrow\rangle = \frac{1}{\sqrt{6}\sqrt{3}}\Big[& 2|u\uparrow u\uparrow d\downarrow\rangle + 2|u\uparrow d\downarrow u\uparrow\rangle + \\
& 2|d\downarrow u\uparrow u\uparrow\rangle - |u\uparrow u\downarrow d\uparrow\rangle - \\
& |u\uparrow d\uparrow u\downarrow\rangle - |d\uparrow u\uparrow u\downarrow\rangle - \\
& |u\downarrow u\uparrow d\uparrow\rangle - |u\downarrow d\uparrow u\uparrow\rangle - \\
& |d\uparrow u\downarrow u\uparrow\rangle \Big]
\end{aligned}
$$

References

- An Introduction to Quarks and Partons by F. E. Close
- A Modern Introduction to Particle Physics by Fayyazuddin and Riazuddin
- Particle Physics: A Comprehensive Introduction by Abraham Seiden
- Class Notes, Physics 221A, Bruce Schumm
- Class Notes, Physics 251, Howie Haber