DUE: Thursday April 18, 2013

1. Consider the set $\mathbb{R}^2$ consisting of pairs of real numbers. For $(x, y) \in \mathbb{R}^2$, define scalar multiplication by: $c(x, y) = (cx, cy)$ for any real number c, and define vector addition and multiplication as follows:

$$\begin{aligned}(x_1, y_1) + (x_2, y_2) &= (x_1 + x_2, y_1 + y_2)\,,\\ (x_1, y_1) \cdot (x_2, y_2) &= (x_1 x_2, y_1 y_2)\,.\end{aligned}$$

(a) Is $\mathbb{R}^2$ a group?

(b) Is $\mathbb{R}^2$ a field?

(c) Is $\mathbb{R}^2$ a linear vector space (over $\mathbb{R}$)?

(d) Is $\mathbb{R}^2$ a linear algebra (over $\mathbb{R}$)?

2. Consider the following two groups:

$T = \{\text{proper rotations that map a regular tetrahdron into itself}\}$,

$T_d = \{\text{proper rotations and reflections that map a regular tetrahdron into itself}\}$.

Show that the following isomorphisms are valid: $T \cong A_4$ and $T_d \cong S_4$.

3. Consider the dihedral group D_4.

(a) Write down the group multiplication table.

(b) Enumerate the subgroups, the normal subgroups and the classes.

(c) Identify the factor groups. Is the full group the direct product of some of its subgroups?

(d) Write out the class multiplication table.

(e) Determine explicitly the matrices of the regular representation.

(f) Write out an explicit irreducible two-dimensional representation of D_4. Check that the group multiplication table is preserved. Verify that this representation is irreducible.

4. The *center* of a group G, denoted by $Z(G)$, is defined as the set of elements $z \in G$ that commute with all elements of the group. That is,

$$Z(G) = \{z \in G \,|\, zg = gz\,, \forall\, g \in G\}\,.$$

(a) Show that $Z(G)$ is an abelian subgroup of G.

(b) Show that $Z(G)$ is a normal subgroup of G.

(c) Find the center of D_4 and construct the group $D_4/Z(D_4)$. Determine whether the isomorphism $D_4 \cong [D_4/Z(D_4)] \otimes Z(D_4)$ is valid.

5. An automorphism is defined as an isomorphism of a group G onto itself.

(a) Show that for any $g \in G$, the mapping $T_g(x) = gxg^{-1}$ is an automorphism (called an *inner automorphism*), where $x \in G$.

(b) Show that the set of all inner automorphisms of G, denoted by $\mathcal{I}(G)$, is a group.

(c) Show that $\mathcal{I}(G) \simeq G/Z(G)$, where $Z(G)$ is the center of G.

(d) Show that the set of all automorphisms of G, denoted by $\mathcal{A}(G)$, is a group and that $\mathcal{I}(G)$ is an invariant subgroup. (The factor group $\mathcal{A}(G)/\mathcal{I}(G)$ is called the group of *outer automorphisms* of G.)

(e) Illustrate these results for $G = S_3$ and $G = \mathbb{Z}$.

6. Consider an arbitrary orthogonal matrix R, which satisfies $RR^{\mathsf{T}} = \mathbb{1}$ (where $\mathbb{1}$ is the identity matrix).

(a) Prove that the possible values of $\det R$ are ± 1. [*HINT:* Consider $\det(RR^{\mathsf{T}})$ and use one of the well-known properties of determinants.]

(b) The group SO(2) consists of all 2×2 orthogonal matrices with unit determinant. Prove that SO(2) is an abelian group.

(c) The group O(2) consists of all 2×2 orthogonal matrices, with no restriction on the sign of its determinant. Is O(2) abelian or non-abelian? (If the latter, exhibit two O(2) matrices that do not commute.)