DUE: THURSDAY, JUNE 13, 2013

FINAL PROJECTS ALERT: The presentations of the final projects will take place during final exams week. The slides from your presentation will be posted to the class website.

1. This problem concerns the Lie group SO(4) and its Lie algebra $\mathfrak{so}(4)$.

(a) Work out the Lie algebra $\mathfrak{so}(4)$ and verify that $\mathfrak{so}(4) \cong \mathfrak{so}(3) \oplus \mathfrak{so}(3)$.

HINT: Show that there is a choice of basis for $\mathfrak{so}(4)$ consisting of 4×4 antisymmetric matrices that contain precisely two non-zero entries: 1 and -1. Evaluate the commutation relations of these $\mathfrak{so}(4)$ generators. Then, by choosing a new basis consisting of sums and differences of pairs of the old $\mathfrak{so}(4)$ generators, show that the resulting commutation relations are isomorphic to the commutation relations of the Lie algebra $\mathfrak{so}(3) \oplus \mathfrak{so}(3)$.

(b) What is the universal covering group of SO(4)? What is the center of SO(4)? What is the adjoint group corresponding to the $\mathfrak{so}(4)$ Lie algebra?

(c) Calculate the Killing form of $\mathfrak{so}(4)$ and verify that this Lie algebra is semisimple and compact.

2. A Lie algebra $\mathfrak{g}$ is defined by the commutation relations of the generators,

$$[e_a, e_b] = f_{ab}^{c} e_c \,.$$

Consider the finite-dimensional matrix representations of the e_a. We shall denote the corresponding generators in the adjoint representation by F_a and in an arbitrary irreducible representation R by R_a. The dimension of the adjoint representation, d, is equal to the dimension of the Lie algebra $\mathfrak{g}$, while the dimension of R will be denoted by d_R.

(a) Show that the Cartan-Killing metric g_{ab} can be written as $g_{ab} = \mathrm{Tr}(F_a F_b)$.

(b) If $\mathfrak{g}$ is a simple real compact Lie algebra, prove that for any irreducible representation R,

$$\mathrm{Tr}(R_a R_b) = c_R g_{ab} \,,$$

where c_R is called the *index* of the irreducible representation R.

HINT: Choose a basis where g_{ab} is proportional to δ_{ab}. Then the f_{ab}^{c} are antisymmetric in all three indices. Show that $\mathrm{Tr}\,[R_a, R_b]R_c = \mathrm{Tr}\,R_a[R_b, R_c]$ and argue that this implies that $\mathrm{Tr}\,R_a R_b$, viewed as the elements of a $d \times d$ matrix, commutes with all Lie algebra elements in the adjoint representation. Finally, invoke Schur's lemma. (Note that by complexifying the simple real compact Lie algebra, one can easily show that the above result also holds for any simple complex Lie algebra.)

(c) The quadratic Casimir operator is defined as $C_2 \equiv g^{ab} e_a e_b$ where g^{ab} is the inverse of g_{ab}. Recall that C_2 commutes with all elements of the Lie algebra. Hence, by Schur's lemma, C_2 must be a multiple of the identity operator. Let us write $C_2 = C_2(R)\mathbf{I}$ where $\mathbf{I}$ is the $d_R \times d_R$ identity matrix and $C_2(R)$ is the eigenvalue of the Casimir operator in the irreducible representation R. As noted above, d is the dimension of the Lie algebra $\mathfrak{g}$. Show that $C_2(R)$ is related to the index c_R by

$$C_2(R) = \frac{d c_R}{d_R}\,.$$

Check this formula in the case that R is the adjoint representation.

HINT: The matrix elements of the R_a are $(R_a)_{ij}$, where $i, j = 1, \ldots, d_R$. If you keep the matrix element indices explicit, then the derivation of the above result is straightforward.

(d) Compute the index of an arbitrary irreducible representation of $\mathfrak{su}(2)$.

(e) Compute the index of the defining representation of $\mathfrak{su}(3)$ and generalize this result to $\mathfrak{su}(n)$.

3. Consider the simple Lie algebra generated by the ten 4×4 matrices: $\sigma_a \otimes \mathbf{I}$, $\sigma_a \otimes \tau_1$, $\sigma_a \otimes \tau_3$ and $\mathbf{I} \otimes \tau_2$, where $(\mathbf{I}, \sigma_a)$ and $(\mathbf{I}, \tau_a)$ are the 2×2 identity and Pauli matrices in orthogonal spaces. For example, since $\tau_3 = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$, we obtain in block matrix form:

$$\sigma_a \otimes \tau_3 = \left(\begin{array}{c|c} \sigma_a & \mathbf{0} \\ \hline \mathbf{0} & -\sigma_a \end{array}\right)\,, \qquad (a = 1, 2, 3)\,,$$

where $\mathbf{0}$ is the 2×2 zero matrix. The remaining seven matrices can be likewise obtained. Take $H_1 = \sigma_3 \otimes \mathbf{I}$ and $H_2 = \sigma_3 \otimes \tau_3$ as the Cartan subalgebra. Note that if A, B, C, and D are 2×2 matrices, then $(A \otimes B)(C \otimes D) = AC \otimes BD$.

(a) Find the roots. Normalize the roots such that the shortest root vector has length 1.

(b) Determine the simple roots and evaluate the corresponding Cartan matrix. Deduce the Dynkin diagram for this Lie algebra and identify it by name.

(c) The fundamental weights $\boldsymbol{m}_i$ are defined such that

$$\frac{2(\boldsymbol{m}_i, \boldsymbol{\alpha}_j)}{(\boldsymbol{\alpha}_j, \boldsymbol{\alpha}_j)} = \delta_{ij}\,, \qquad \text{for } i, j = 1, 2, \ldots, r\,,$$

where $\boldsymbol{\alpha}_j \in \Pi$ (the set of simple roots) and r is the rank of the Lie algebra. Using the results of part (b), determine the fundamental weights.

HINT: Expand the $\boldsymbol{m}_i$ as a linear combination of the simple roots and solve for the coefficients.