1. This problem concerns the Lie group SO(4) and its Lie algebra $\mathfrak{so}(4)$.

   (a) Work out the Lie algebra $\mathfrak{so}(4)$ and verify that $\mathfrak{so}(4) \cong \mathfrak{so}(3) \oplus \mathfrak{so}(3)$.

The defining representation of the Lie algebra $\mathfrak{so}(n)$ is

$$\mathfrak{so}(n) = \{\mathcal{A} \,|\, \mathcal{A} \in \mathfrak{gl}(n, \mathbb{R}) \text{ such that } \mathcal{A}^{\mathsf{T}} = -\mathcal{A}\}\,,$$

where $\mathfrak{gl}(n, \mathbb{R})$ is the set of all real $n \times n$ matrices. Recall that a suitable basis for the defining representation of $\mathfrak{so}(3)$, which consists of all $3 \times 3$ real antisymmetric matrices, is $(\mathcal{A}_i)_{jk} = -\epsilon_{ijk}$, where $i$, $j$ and $k$ can take on the values 1, 2 and 3. To find a suitable basis for the defining representation of $\mathfrak{so}(4)$, one can generalize the $\mathcal{A}_i$ of $\mathfrak{so}(3)$ by choosing

$$(\mathcal{A}_i)_{jk} = \left(\begin{array}{ccc|c} & & & 0 \\ & -\epsilon_{ijk} & & 0 \\ & & & 0 \\ \hline 0 & 0 & 0 & 0 \end{array}\right), \qquad \text{where } i, j, k = 1, 2, 3\,. \tag{1}$$

Since a $4 \times 4$ real antisymmetric matrix has six independent parameters, we need to choose three additional linearly-independent antisymmetric matrices to complete the basis for $\mathfrak{so}(4)$. We therefore introduce three antisymmetric matrices $\mathcal{B}_i$ by placing a 1 in one of the non-diagonal elements of the fourth row (and a corresponding $-1$ required by the antisymmetry property of the matrix), with all other elements zero. That is, a suitable basis for $\mathfrak{so}(4)$ is given by:

$$\mathcal{A}_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad \mathcal{A}_2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad \mathcal{A}_3 = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\mathcal{B}_1 = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \qquad \mathcal{B}_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \qquad \mathcal{B}_3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

One can easily verify that the six generators of $\mathfrak{so}(4)$ satisfy the following commutation relations:

$$[\mathcal{A}_i\,,\,\mathcal{A}_j] = \epsilon_{ijk}\mathcal{A}_k\,, \qquad [\mathcal{B}_i\,,\,\mathcal{B}_j] = \epsilon_{ijk}\mathcal{A}_k\,, \qquad [\mathcal{A}_i\,,\,\mathcal{B}_j] = \epsilon_{ijk}\mathcal{B}_k\,. \tag{2}$$

Note that the commutation relations satisfied by the $\mathcal{A}_i$ are precisely those of $\mathfrak{so}(3)$, which is not surprising in light of eq. (1).

The form of the commutators given in eq. (2) is not completely transparent. To understand the implications of eq. (2), it is convenient to define a new set of $\mathfrak{so}(4)$ generators that are real linear combinations of the $\mathcal{A}_i$ and $\mathcal{B}_i$. Thus, we define,

$$X_i \equiv \tfrac{1}{2}(\mathcal{A}_i + \mathcal{B}_i)\,, \qquad Y_i \equiv \tfrac{1}{2}(\mathcal{A}_i - \mathcal{B}_i)\,, \qquad \text{where } i = 1,2,3\,. \tag{3}$$

Using eq. (2), it is a simple matter to work out the commutation relations among the $X_i$ and $Y_i$:

$$[X_i\,,\,X_j] = \epsilon_{ijk}X_k\,, \qquad [Y_i\,,\,Y_j] = \epsilon_{ijk}Y_k\,, \qquad [X_i\,,\,Y_j] = 0\,. \tag{4}$$

Thus, we have succeeding in writing the $\mathfrak{so}(4)$ commutation relations in such a way that the generators $\{X_i\}$ and $\{Y_i\}$ are decoupled. In particular, the $\{X_i\}$ and $\{Y_i\}$ each satisfy $\mathfrak{so}(3)$ commutation relations. Hence, $\mathfrak{so}(4)$ is a direct sum of two independent $\mathfrak{so}(3)$ Lie algebras. That is,[1]

$$\mathfrak{so}(4) \cong \mathfrak{so}(3) \oplus \mathfrak{so}(3)\,. \tag{5}$$

(b) What is the universal covering group of SO(4)? What is the center of SO(4)? What is the adjoint group corresponding to the $\mathfrak{so}(4)$ Lie algebra?

Since the universal covering group of SO(3) is SU(2), we can use eq. (5) to conclude that the universal covering group of SO(4) is SU(2)⊗SU(2).[2] In particular,

$$\mathrm{SO}(4) \cong \mathrm{SU}(2) \otimes \mathrm{SU}(2)/\mathbb{Z}_2\,. \tag{6}$$

To justify eq. (6), consider the centers of SO(4) and SU(2)⊗SU(2). The center of SO(4) consists of all orthogonal matrices of unit determinant that are multiples of the identity. There are only two such matrices, $\mathbb{1}_{4\times4}$ and $-\mathbb{1}_{4\times4}$, where $\mathbb{1}_{4\times4}$ is the $4\times4$ identity matrix. Hence,

$$Z(\mathrm{SO}(4)) = \mathbb{Z}_2\,.$$

The center of SU(2) is $\{\mathbb{1}_{2\times2}\,,-\mathbb{1}_{2\times2}\} \cong \mathbb{Z}_2$ so that

$$Z(\mathrm{SU}(2) \otimes \mathrm{SU}(2)) = \mathbb{Z}_2 \otimes \mathbb{Z}_2\,.$$

Thus, only one $\mathbb{Z}_2$ factor can appear in eq. (6).

Finally, the adjoint group by definition has a trivial center. Thus, the adjoint group of SO(4) can be expressed in a number of equivalent forms,

$$\mathrm{SO}(4)/\mathbb{Z}_2 \cong \mathrm{SO}(3) \otimes \mathrm{SO}(3) \cong \mathrm{SU}(2) \otimes \mathrm{SU}(2)/\mathbb{Z}_2 \otimes \mathbb{Z}_2\,,$$

where we have made use of the well-known isomorphism, $\mathrm{SO}(3)\cong\mathrm{SU}(2)/\mathbb{Z}_2$. Indeed, SO(3)⊗SO(3) has a trivial center since SO(3) has a trivial center.

[1] Since $\mathfrak{su}(2) \cong \mathfrak{so}(3)$ as Lie algebras, we can equally well write $\mathfrak{so}(4) \cong \mathfrak{su}(2) \oplus \mathfrak{su}(2)$.

[2] Since the Lie group is obtained by exponentiation of the Lie algebra, a direct sum of Lie algebras correspond to a direct product of Lie groups.

(c) Calculate the Killing form of $\mathfrak{so}(4)$ and verify that this Lie algebra is semisimple and compact.

The Cartan-Killing form can be expressed in terms of the Lie algebra structure constants,

$$g_{ab} = f_{ac}^{d} f_{bd}^{c} \,. \tag{7}$$

In this expression, the indices $a$, $b$, $c$ and $d$ range over $1, 2, \ldots, 6$, corresponding to the six generators of $\mathfrak{so}(4)$. It is easiest to evaluate $g_{ab}$ in the basis $\{X_i\,,\,Y_j\}$ [cf, eqs. (3) and (4)]. In this basis,

$$f_{ac}^{d} = \begin{cases} \epsilon_{ijk}\,, & \text{for } a = i\,, b = j\,, \text{ and } d = k\,, \\ \epsilon_{ijk}\,, & \text{for } a = i+3\,, b = j+3\,, \text{ and } d = k+3\,, \\ 0\,, & \text{otherwise}\,, \end{cases}$$

where $i$, $j$ and $k$ range over 1, 2 and 3. Plugging into eq. (7) yields

$$g_{ab} = -2\delta_{ab}\,,$$

which indicates that $\mathfrak{so}(4)$ is a semi-simple and compact Lie algebra.

2. A Lie algebra $\mathfrak{g}$ is defined by the commutation relations of the generators,

$$[e_a, e_b] = f_{ab}^{c} e_c \,.$$

Consider the finite-dimensional matrix representations of the $e_a$. We shall denote the corresponding generators in the adjoint representation by $F_a$ and in an arbitrary irreducible representation $R$ by $R_a$. The dimension of the adjoint representation, $d$, is equal to the dimension of the Lie algebra $\mathfrak{g}$, while the dimension of $R$ will be denoted by $d_R$.

(a) Show that the Cartan-Killing metric $g_{ab}$ can be written as $g_{ab} = \mathrm{Tr}(F_a F_b)$.

The Cartan-Killing metric can be expressed in terms of the structure constants as follows,

$$g_{ij} = f_{i\ell}^{k} f_{jk}^{\ell} \,.$$

On the other hand, matrix elements of the adjoint representation are given by:

$$(F_i)^j{}_k = f_{ik}^{j} \,.$$

Therefore,

$$\mathrm{Tr}(F_i F_j) = (F_i)^k{}_\ell (F_j)^\ell{}_k = f_{i\ell}^{k} f_{jk}^{\ell} = g_{ij} \,.$$

(b) If $\mathfrak{g}$ is a simple real compact Lie algebra, prove that for any irreducible representation $R$,

$$\mathrm{Tr}(R_a R_b) = c_R g_{ab}\,,$$

where $c_R$ is called the *index* of the irreducible representation $R$.

Consider a $d$-dimensional Lie algebra $\mathfrak{g}$, whose generators are represented by the matrices $R_a$. These matrices satisfy the Lie algebra commutation relations,

$$[R_a, R_b] = f_{ab}^{c} R_c\,, \qquad \text{where } a, b, c = 1, 2, \ldots, d\,. \tag{8}$$

We first note the following identity:

$$\mathrm{Tr}\big\{[R_a, R_b]R_c\big\} = \mathrm{Tr}\big\{R_a[R_b, R_c]\big\}\,. \tag{9}$$

The proof of eq. (9) is straightforward:

$$\begin{aligned}\mathrm{Tr}\big\{[R_a, R_b]R_c\big\} &= \mathrm{Tr}\big\{(R_aR_b - R_bR_a)R_c\big\} = \mathrm{Tr}(R_aR_bR_c) - \mathrm{Tr}(R_bR_aR_c)\\ &= \mathrm{Tr}(R_aR_bR_c) - \mathrm{Tr}(R_aR_cR_b) = \mathrm{Tr}\big\{R_a(R_bR_c - R_cR_b)\big\} = \mathrm{Tr}\big\{R_a[R_b, R_c]\big\}\,,\end{aligned}$$

after using the cyclic properties of the trace. Making use of eq. (8) in eq. (9) yields:

$$f_{ab}^{d}\,\mathrm{Tr}(R_dR_c) = f_{bc}^{d}\,\mathrm{Tr}(R_aR_d)\,. \tag{10}$$

To make further progress, recall that $f_{abc} \equiv g_{ad} f_{bc}^{d}$ is totally antisymmetric under the interchange of any pair of indices $a$, $b$ and $c$. It follows that

$$f_{bc}^{d} = g^{ad} f_{abc}\,, \tag{11}$$

where $g^{ad}$ is the inverse Cartan metric tensor. It is convenient to multiply both sides of eq. (10) by $g^{ea}$ to obtain:

$$g^{ea} f_{ab}^{d}\,\mathrm{Tr}(R_dR_c) = g^{ea} f_{bc}^{d}\,\mathrm{Tr}(R_aR_d)\,. \tag{12}$$

Using eq. (11) and the antisymmetry properties of $f_{abh}$,

$$g^{ea} f_{ab}^{d} = g^{ea} g^{hd} f_{hab} = g^{ea} g^{hd} f_{abh} = g^{hd} f_{bh}^{e}\,.$$

Inserting this result into eq. (12) yields

$$g^{hd} f_{bh}^{e}\,\mathrm{Tr}(R_dR_c) = g^{ea} f_{bc}^{d}\,\mathrm{Tr}(R_aR_d)\,. \tag{13}$$

Consider the $d \times d$ matrix whose matrix elements are

$$A^{h}{}_{c} \equiv g^{hd}\,\mathrm{Tr}(R_dR_c)\,. \tag{14}$$

We can then rewrite eq. (13) in the following form:

$$f_{bh}^{e} A^{h}{}_{c} = f_{bc}^{d} A^{e}{}_{d}\,. \tag{15}$$

We recognize $f_{bh}^{e} = (F_b)^e{}_h$ and $f_{bc}^{d} = (F_b)^d{}_c$. Hence, eq. (15) is equivalent to the $ec$ component of the matrix equation,

$$F_b A = A F_b\,,$$

for all $b = 1, 2, \ldots, d$.

We proved in class that the adjoint representation of a simple Lie algebra (whose generators are represented by the matrices $F_b$) is irreducible. Applying Schur's second lemma to representations of Lie algebras,[3] any matrix that commutes with all the $F_b$ must be a multiple of the identity. Hence, $A = c\mathbf{I}$ or equivalently.

$$g^{ed}\,\mathrm{Tr}(R_d R_c) = c_R \delta_c^e\,,$$

where $c_R$ is some complex constant. Using $g^{ed}g_{eh} = \delta_h^d$, it immediately follows that

$$\mathrm{Tr}(R_h R_c) = c_R g_{hc}\,, \tag{16}$$

which is the desired result.

(c) The quadratic Casimir operator is defined as $C_2 \equiv g^{ab}e_a e_b$ where $g^{ab}$ is the inverse of $g_{ab}$. Recall that $C_2$ commutes with all elements of the Lie algebra. Hence, by Schur's lemma, $C_2$ must be a multiple of the identity operator. Let us write $C_2 = C_2(R)\mathbf{I}$, where $\mathbf{I}$ is the $d_R \times d_R$ identity matrix and $C_2(R)$ is the eigenvalue of the Casimir operator in the irreducible representation $R$. As noted above, $d$ is the dimension of the Lie algebra $\mathfrak{g}$. Show that $C_2(R)$ is related to the index $c_R$ by

$$C_2(R) = \frac{dc_R}{d_R}\,.$$

Check this formula in the case that $R$ is the adjoint representation.

By definition,

$$C_2(R)\mathbf{I} = g^{ab}R_a R_b\,, \tag{17}$$

where $\mathbf{I}$ is the $d_R \times d_R$ identity matrix, $d_R$ is the dimension of the representation $R$, and $a, b = 1, 2, \ldots, d$. Taking the trace of eq. (17) and using eq. (16), it follows that:

$$d_R C_2(R) = g^{ab}\mathrm{Tr}(R_a R_b) = c_R g^{ab} g_{ab} = c_R d\,,$$

since $g^{ab}g_{ab} = \delta_a^a = d$. Hence, solving for $C_2(R)$, one obtains:

$$C_2(R) = \frac{dc_R}{d_R}\,. \tag{18}$$

[3] A review of the proof given in class of Schur's lemmas (which were applied to group representations) reveals that it also applies to representations of Lie algebras. Indeed, for any algebraic structure $\mathscr{A}$, Schur's second lemma states that if there exists a matrix $M$ such that $D(\mathcal{A})M = MD(\mathcal{A})$ for *all* $\mathcal{A} \in \mathscr{A}$, where $D(\mathcal{A})$ is an $n$-dimensional irreducible matrix representation of $\mathcal{A}$ (over a complex representation space $\mathbb{C}^n$), then it follows that $M$ must be a multiple of the identity matrix. In particular, any element of a Lie algebra $\mathscr{A}$ can be expressed as some linear combination of the the generators $\mathcal{A}_a$ (which serve as a basis for the Lie algebra). Consequently, if $D(\mathcal{A}_a)M = MD(\mathcal{A}_a)$ for all $a = 1, 2, \ldots, d$, then it follows that $D(\mathcal{A})M = MD(\mathcal{A})$ for all $\mathcal{A} \in \mathscr{A}$, and Schur's second lemma applies.

For the adjoint representation (usually denoted by $R = A$), we have $d_A = d$. Moreover, the adjoint representation generators are $(R_a)^b{}_c = f_{ac}^{\ b}$, as shown in class. Hence,

$$\mathrm{Tr}(R_a R_d) = (R_a)^b{}_c (R_d)^c{}_b = f_{ac}^{\ b} f_{db}^{\ c} = g_{ad}\,,$$

where we used the definition of the Cartan metric tensor at the last step. Comparing this result with that of eq. (16) yields $c_A = 1$. Hence, eq. (18) implies that $C_2(A) = 1$ in agreement with the theorem proven in class.

(d) Compute the index of an arbitrary irreducible representation of $\mathfrak{su}(2)$.

For $\mathfrak{su}(2)$, the irreducible representations are labeled by $j = 0, \frac{1}{2}, 1, \frac{3}{2} \ldots$. The quadratic Casimir operator is proportional to $J_1^2 + J_2^2 + J_3^2$, where $[J_i, J_j] = i\epsilon_{ijk}J_k$ in the physicist convention. Since the eigenvalue of of $J_1^2 + J_2^2 + J_3^2$ is $j(j+1)$, we shall adjust the overall normalization of the Casimir operator so that $C_2(A) = 1$. Given that the adjoint representation of $\mathfrak{su}(2)$ corresponds to $j = 1$, it follows that:

$$C_2(j) = \tfrac{1}{2}j(j+1)\,.$$

We can now use eq. (18) to obtain the index of an irreducible representation of $\mathfrak{su}(2)$. Using $d_R = 2j+1$ for the irreducible representation labeled by $j$, it follow that the index $c_R$ is given by

$$c(j) = \tfrac{1}{6}j(j+1)(2j+1)\,.$$

In the defining representation, $j = \frac{1}{2}$, and we find $c_F \equiv c(\frac{1}{2}) = \frac{1}{4}$. In the adjoint representation, $j = 1$ and we find that $c_A \equiv c(1) = 1$ as expected from part (b).

(e) Compute the index of the defining representation of $\mathfrak{su}(3)$ and generalize this result to $\mathfrak{su}(n)$.

First, consider the Lie algebra $\mathfrak{su}(3)$. We choose the generators in the defining representation to be the Gell-Mann matrices, $\frac{1}{2}\lambda_a$. Following the mathematician's conventions, we define $T_a \equiv -\frac{1}{2}i\lambda_a$ so that

$$[T_a, T_b] = f_{abc}T_c\,,$$

where the $f_{abc}$ are the totally antisymmetric structure constants in the convention where the $T_a$ satisfy

$$\mathrm{Tr}(T_a T_b) = -\tfrac{1}{4}\mathrm{Tr}(\lambda_a\lambda_b) = -\tfrac{1}{2}\delta_{ab}\,, \tag{19}$$

using the explicit form for the Gell-Mann matrices displayed in class. In this basis choice,

$$g_{ab} = f_{ad}^{\ c} f_{bd}^{\ c} = -3\delta_{ab}\,,$$

using the explicit form for the $\mathfrak{su}(3)$ structure constants listed in the class handout on SU(3). The index of the defining representation, usually denoted by $c_F$ (since physicists also refer to this representation as the fundamental representation), can be obtained from eq. (16),

$$\mathrm{Tr}(T_a T_b) = c_F(-3\delta_{ab})\,.$$

Using eq. (19) to compute the trace, we end up with

$$c_F = \tfrac{1}{6}\,.$$

To generalize these results to $\mathfrak{su}(n)$, we shall make use of the construction of the $\mathfrak{su}(n)$ Lie algebra given in class. There, we defined traceless $n \times n$ Hermitian matrices,

$$(F^a{}_b)_{cd} = \delta_{bc}\delta^a_d - \frac{1}{n}\delta^a_b\delta_{cd}\,,$$

which satisfy the commutation relations,

$$[F^a{}_b, F^c{}_d] = \delta^a_d F^c{}_b - \delta^c_b F^a{}_d\,. \tag{20}$$

The generalized Gell-Mann matrices are:

$$\lambda_1 = F^1{}_2 + F^2{}_1 = \left(\begin{array}{c|c} \sigma_1 & 0 \\ \hline 0 & 0 \end{array}\right), \qquad \lambda_2 = i(F^1{}_2 - F^2{}_1) = \left(\begin{array}{c|c} \sigma_2 & 0 \\ \hline 0 & 0 \end{array}\right),$$

$$\lambda_3 = F^1{}_1 - F^2{}_2 = \left(\begin{array}{c|c} \sigma_3 & 0 \\ \hline 0 & 0 \end{array}\right), \qquad \text{etc.} \tag{21}$$

where the Pauli matrices occupy the upper left $2 \times 2$ block of the $n \times n$ matrix generators (with all other elements zero). In the mathematician's convention, we define $T_a = -\frac{1}{2}i\lambda_a$ and $[T_a, T_b] = f_{abc}T_c$, where the $f_{abc}$ are totally antisymmetric and $\mathrm{Tr}(T_aT_b) \propto \delta_{ab}$. To compute the constant of proportionality, one can check for example that

$$\mathrm{Tr}(T_3T_3) = -\tfrac{1}{4}\mathrm{Tr}(\lambda_3\lambda_3) = -\tfrac{1}{2}\,,$$

using eq. (21). Clearly, the constant of proportionality does not depend on the choice of $a$ and $b$. Hence, it follows that the generators of $\mathfrak{su}(n)$ in the defining representation satisfy

$$\mathrm{Tr}(T_aT_b) = -\tfrac{1}{2}\delta_{ab}\,. \tag{22}$$

Next, we evaluate the Cartan metric tensor, which is given by:

$$g_{ab} = f^c_{ad}f^d_{bc}\,. \tag{23}$$

In the convention where the generators satisfy $\mathrm{Tr}(T_aT_b) \propto \delta_{ab}$, the Cartan metric tensor also satisfies $g_{ab} \propto \delta_{ab}$, in light of eq. (16). To determine the proportionality constant, consider

$$[T_3, T_c] = f_{3cd}T_d\,.$$

We can evaluate $g_{33} = f_{3dc}f_{3cd}$ by examining eq. (20). In particular,

$$[T_3, F^2{}_1] = F^2{}_1\,, \qquad [T_3, F^1{}_2] = -F^1{}_2\,, \qquad [T_3, F^a{}_1] = \tfrac{1}{2}F^a{}_1\,, \qquad [T_3, F^1{}_a] = -\tfrac{1}{2}F^1{}_a\,,$$

$$[T_3, F^a{}_2] = -\tfrac{1}{2}F^a{}_1\,, \quad [T_3, F^2{}_a] = \tfrac{1}{2}F^1{}_a\,, \qquad [T_3, F^a{}_b] = [T_3, F^b{}_a] = 0\,, \tag{24}$$

for $a \neq b$ and $a, b = 3, 4, \ldots, n$. Note that the non-diagonal generators $T_c$ of the form ${F^a}_b + {F^b}_a$ and $i({F^a}_b - {F^b}_a)$ for $a < b$ with $a = 1$ or $a = 2$ are the only generators that do not commute with $T_3$. Eq. (24) provides the necessary information to evaluate $g_{33}$,

$$g_{33} = (+1)(-1) + (n-1)\left(\tfrac{1}{2}\right)\left(-\tfrac{1}{2}\right) = -n\,.$$

where the first term on the right-hand side derives from $f_{312}f_{321}$, whereas the remaining terms derive from the remaining combination of non-zero structure constants. That is,

$$g_{ab} = f_{ad}^{c} f_{bc}^{d} = -n\delta_{ab}\,.$$

The index of the defining representation can be obtained from eq. (16),

$$\mathrm{Tr}(T_a T_b) = c_F(-n\delta_{ab})\,.$$

Using eq. (22) to compute the trace, we end up with

$$c_F = \frac{1}{2n}\,. \tag{25}$$

One sees that this general result is consistent with the corresponding results of $\mathfrak{su}(2)$ and $\mathfrak{su}(3)$ previously obtained.

**Remarks:**

Using eqs. (18) and (25), one can compute the eigenvalue of the quadratic Casimir operator in the defining representation of $\mathfrak{su}(n)$. In particular, since $d = n^2 - 1$, $d_F = n$ and $c_F = 1/(2n)$, it follows that:

$$C_2(F) = \frac{n^2-1}{2n^2}\,.$$

Moreover, the Casimir operator in the defining representation of $\mathfrak{su}(n)$ is given by

$$C_2(A) = 1\,,$$

according to the theorem proved in class. However, note that the Casimir operator of $\mathfrak{su}(n)$ is defined in an arbitrary irreducible representation $R$ by

$$C_2 = g^{ab} R_a R_b = -\frac{1}{n}\sum_{a=1}^{n^2-1} R_a R_a\,, \tag{26}$$

where we have used eq. (23) [recall that $g^{ab}$ is the inverse of $g_{ab}$]. In the physics literature, in the case of $\mathfrak{su}(n)$ one typically defines $C_2$ by omitting the overall factor of $1/n$ in eq. (26). Consequently, $C_2(R)$ is a factor of $n$ larger than indicated above, in which case

$$C_2(F) = \frac{n^2-1}{2n}\,, \qquad\qquad C_a(A) = n\,.$$

Additional details on the Casimir operator and index of an irreducible representation of a simple Lie algebra can be found in the class handout entitled, *The eigenvalues of the quadratic Casimir operator and second-order indices of a simple Lie algebra.*

3. Consider the simple Lie algebra generated by the ten $4\times 4$ matrices: $\sigma_a\otimes\mathbf{I}$, $\sigma_a\otimes\tau_1$, $\sigma_a\otimes\tau_3$ and $\mathbf{I}\otimes\tau_2$, where $(\mathbf{I},\sigma_a)$ and $(\mathbf{I},\tau_a)$ are the $2\times 2$ identity and Pauli matrices in orthogonal spaces. For example, since $\tau_3=\left(\begin{smallmatrix}1 & 0\\ 0 & -1\end{smallmatrix}\right)$, we obtain in block matrix form:

$$\sigma_a\otimes\tau_3=\left(\begin{array}{c|c}\sigma_a & \mathbf{0}\\ \hline \mathbf{0} & -\sigma_a\end{array}\right),\qquad (a=1,2,3)\,,$$

where $\mathbf{0}$ is the $2\times 2$ zero matrix. The remaining seven matrices can be likewise obtained. Take $H_1=\sigma_3\otimes\mathbf{I}$ and $H_2=\sigma_3\otimes\tau_3$ as the Cartan subalgebra. Note that if $A$, $B$, $C$, and $D$ are $2\times 2$ matrices, then $(A\otimes B)(C\otimes D)=AC\otimes BD$.

(a) Find the roots. Normalize the roots such that the shortest root vector has length 1.

First, we write out the ten generators explicitly in block matrix form:

$$A_a\equiv\sigma_a\otimes\tau_1=\left(\begin{array}{c|c}\mathbf{0} & \sigma_a\\ \hline \sigma_a & \mathbf{0}\end{array}\right),\qquad (a=1,2,3)\,,$$

$$B_a\equiv\sigma_a\otimes\tau_3=\left(\begin{array}{c|c}\sigma_a & \mathbf{0}\\ \hline \mathbf{0} & -\sigma_a\end{array}\right),\qquad (a=1,2,3)\,,$$

$$C_a\equiv\sigma_a\otimes\mathbf{I}=\left(\begin{array}{c|c}\sigma_a & \mathbf{0}\\ \hline \mathbf{0} & \sigma_a\end{array}\right),\qquad (a=1,2,3)\,,$$

$$D\equiv\mathbf{I}\otimes\tau_2=\left(\begin{array}{c|c}\mathbf{0} & -i\mathbf{I}\\ \hline i\mathbf{I} & \mathbf{0}\end{array}\right).\tag{27}$$

To check that these generators actually generate a Lie algebra, we work out all the commutation relations:

$$\begin{array}{lll}[A_a,A_b]=2i\epsilon_{abc}C_c\,, & [B_a,B_b]=2i\epsilon_{abc}C_c\,, & [C_a,C_b]=2i\epsilon_{abc}C_c\,,\\ [A_a,B_b]=-2i\delta_{ab}D\,, & [A_a,C_b]=2i\epsilon_{abc}A_c\,, & [B_a,C_b]=2i\epsilon_{abc}B_c\,,\\ [A_a,D]=2iB_a\,, & [B_a,D]=-2iA_a\,, & [C_a,D]=0\,,\end{array}\tag{28}$$

where we have used $\sigma_a\sigma_b=\mathbf{I}\delta_{ab}+i\epsilon_{abc}\sigma_c$. For example,

$$\begin{aligned}[A_a,B_b]=A_aB_b-B_bA_a&=\left(\begin{array}{c|c}\mathbf{0} & \sigma_a\\ \hline \sigma_a & \mathbf{0}\end{array}\right)\left(\begin{array}{c|c}\sigma_b & \mathbf{0}\\ \hline \mathbf{0} & -\sigma_b\end{array}\right)-\left(\begin{array}{c|c}\sigma_b & \mathbf{0}\\ \hline \mathbf{0} & -\sigma_b\end{array}\right)\left(\begin{array}{c|c}\mathbf{0} & \sigma_a\\ \hline \sigma_a & \mathbf{0}\end{array}\right)\\ &=\left(\begin{array}{c|c}\mathbf{0} & -(\sigma_a\sigma_b+\sigma_b\sigma_a)\\ \hline \sigma_a\sigma_b+\sigma_b\sigma_a & \mathbf{0}\end{array}\right)\\ &=\left(\begin{array}{c|c}\mathbf{0} & -2\mathbf{I}\delta_{ab}\\ \hline 2\mathbf{I}\delta_{ab} & \mathbf{0}\end{array}\right)=-2i\delta_{ab}D\,.\end{aligned}\tag{29}$$

Alternatively, one can derive the commutation relations displayed in eq. (28) by employing the direct product representation of the Lie algebra generators given in eq. (27)

and using $(A \otimes B)(C \otimes D) = AC \otimes BD$. For example, eq. (29) can also be obtained as follows.

$$
\begin{aligned}
[A_a, B_b] &= (\sigma_a \otimes \tau_1)(\sigma_b \otimes \tau_3) - (\sigma_b \otimes \tau_3)(\sigma_a \otimes \tau_1) \\
&= (\sigma_a \sigma_b) \otimes (\tau_1 \tau_3) - (\sigma_b \sigma_a) \otimes (\tau_3 \tau_1) \\
&= (\sigma_a \sigma_b) \otimes (-i\tau_2) - (\sigma_b \sigma_a) \otimes (i\tau_2) \\
&= (\sigma_a \sigma_b + \sigma_b \sigma_a) \otimes (-i\tau_2) \\
&= (2\mathbf{I}\delta_{ab}) \otimes (-i\tau_2) = -2i\delta_{ab}\,\mathbf{I} \otimes \tau_2 = -2i\delta_{ab} D\,.
\end{aligned}
$$

All other commutation relations are easily derived using either of the methods shown above. Thus, the ten generators $\{A_a, B_a, C_a, D\}$ generate a Lie algebra, since the commutation relations close.

To determine the roots, we take $H_1 = \sigma_3 \otimes \mathbf{I} = C_3$ and $H_2 = \sigma_3 \otimes \tau_3 = B_3$ to generate the Cartan subalgebra. Indeed, these two generators are diagonal in the representation given in eq. (27). We now rewrite the commutation relations given in eq. (28) in the Cartan-Weyl form. Starting from the commutation relations,

$$
[B_3, A_1] = [B_3, A_2] = 0\,, \qquad [C_3, A_1] = 2iA_2\,, \qquad [C_3, A_2] = -2iA_1\,,
$$

it is clear that we should define $A_\pm \equiv A_1 \pm iA_2$, in which case,

$$
[B_2, A_\pm] = 0\,, \qquad [C_3, A_\pm] = \pm 2A_\pm\,. \tag{30}
$$

Next, we focus on the commutation relations,

$$
[B_3, A_3] = 2iD\,, \qquad [B_3, D] = -2iA_3\,, \qquad [C_3, A_3] = [C_3, D] = 0\,.
$$

These results motivate the definition $D_\pm \equiv A_3 \pm iD$, in which case,

$$
[B_3, D_\pm] = \pm 2D_\pm\,, \qquad [C_3, D_\pm] = 0\,. \tag{31}
$$

The remaining commutation relations are:

$$
[B_3, B_1] = 2iC_2\,, \qquad [B_3, B_2] = -2iC_1\,, \qquad [B_3, C_1] = 2iB_2\,, \qquad [B_3, C_2] = -2iB_1\,, \tag{32}
$$

$$
[C_3, B_1] = 2iB_2\,, \qquad [C_3, B_2] = -2iB_1\,, \qquad [C_3, C_1] = 2iC_2\,, \qquad [C_3, C_2] = -2iC_1\,. \tag{33}
$$

Defining $B_\pm \equiv B_1 \pm iB_2$ and $C_\pm \equiv C_1 \pm iC_2$, eqs. (32) and (33) can be rewritten as:

$$
[B_3, B_\pm] = \pm 2C_\pm\,, \qquad [B_3, C_\pm] = \pm 2B_\pm\,, \qquad [C_3, B_\pm] = \pm 2B_\pm\,, \qquad [C_3, C_\pm] = \pm 2C_\pm\,. \tag{34}
$$

Thus, if we define $F_\pm \equiv B_\pm + C_\pm$ and $G_\pm \equiv B_\pm - C_\pm$, the eq. (34) will be in Cartan-Weyl form,

$$
[B_3, F_\pm] = \pm 2F_\pm\,, \qquad [B_3, F_\pm] = \pm 2F_\pm\,, \qquad [C_3, G_\pm] = \mp 2G_\pm\,, \qquad [C_3, G_\pm] = \pm 2G_\pm\,. \tag{35}
$$

To summarize, eqs. (30), (31) and (35) provide the Cartan-Weyl form for the commutation relations among the generators of the Cartan subalgebra and the off-diagonal generators $E_{\boldsymbol{\alpha}} \equiv \{A_\pm, D_\pm, E_\pm, F_\pm\}$.

The root vectors are defined by the Cartan-Weyl form for the Lie algebra commutation relations, $[H_i, E_{\boldsymbol{\alpha}}] = \alpha_i E_{\boldsymbol{\alpha}}$, for $i = 1, 2, \ldots, r$, where $r$ is the number of diagonal generators (and is equal to the rank of the Lie algebra). In the present example, $r = 2$, $H_1 = C_3$, $H_2 = B_3$ and the off diagonal generators are $E_{\boldsymbol{\alpha}} \equiv \{A_\pm, D_\pm, E_\pm, F_\pm\}$. Hence, we identify the root vectors derived from the non-diagonal generators:

$$A_\pm: \quad \pm(0,2)\,, \qquad\qquad D_\pm: \quad \pm(2,0)\,, \tag{36}$$

$$F_\pm: \quad \pm(2,2)\,, \qquad\qquad G_\pm: \quad \pm(-2,2)\,, \tag{37}$$

where the first entry of the root vector is the eigenvalue of $\mathrm{ad}_{C_3}$ and the second entry of the root vector is the eigenvalue of $\mathrm{ad}_{B_3}$ The Cartan metric can be computed from the formula derived in class,

$$g_{ij} = \sum_{\boldsymbol{\alpha}} \alpha_i \alpha_j\,.$$

From the four root vectors obtained in eqs. (36) and (37), we immediately obtain

$$g_{ij} = 24\delta_{ij}\,. \tag{38}$$

The inverse Cartan metric is $g^{ij} = \frac{1}{24}\delta_{ij}$. One can now define the inner product on the root space,

$$(\boldsymbol{\alpha}, \boldsymbol{\beta}) = g^{ij}\alpha_i\beta_j\,. \tag{39}$$

The squared-length of a root vector $\boldsymbol{\alpha}$ is given by

$$(\boldsymbol{\alpha}, \boldsymbol{\alpha}) = g^{ij}\alpha_i\alpha_j = \sum_{i=1}^{2} \alpha_i\alpha_i\,.$$

It is convenient to redefine the inner product given in eq. (39) by introducing an overall multiplicative positive constant such that the new inner product is Euclidean,

$$(\boldsymbol{\alpha}, \boldsymbol{\beta}) = \sum_i \alpha_i\beta_i\,.$$

Moreover, we are always free to rescale the generators of the Cartan subalgebra (which rescales the root vectors) in such a way that the shortest root vector has length 1. In these conventions, the rescaled roots are given by [cf. eqs. (36) and (37)]:

$$\pm(0,1)\,, \quad \pm(1,0)\,, \quad \pm(1,1)\,, \quad \pm(-1,1)\,.$$

and the corresponding root diagram is shown in Fig. 1, which we recognize as the root diagram for the rank-2 Lie algebra $\mathfrak{sp}(2) \cong \mathfrak{so}(5)$.[4]

[4]In the notation used here, $\mathfrak{sp}(n)$ is a Lie algebra of rank $n$. However, many books denote this Lie algebra by $\mathfrak{sp}(2n)$. Both conventions are common in the mathematics and physics literature.

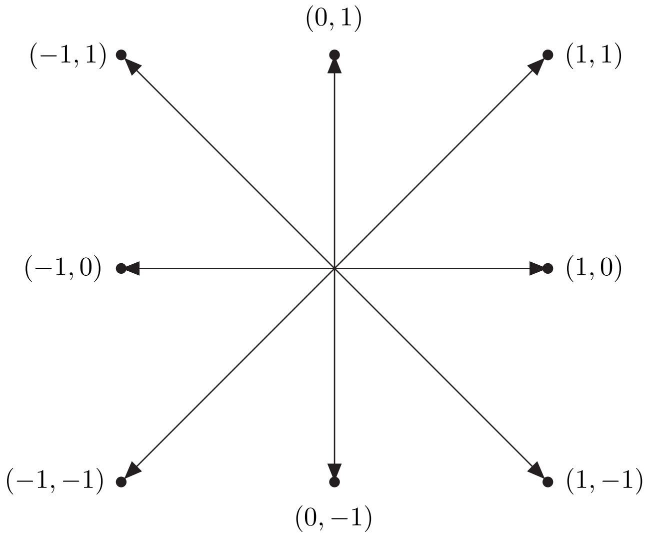


**Figure 1:** The root diagram for $\mathfrak{sp}(2) \cong \mathfrak{so}(5)$.

(b) Determine the simple roots and evaluate the corresponding Cartan matrix. Deduce the Dynkin diagram for this Lie algebra and identify it by name.

The simple roots correspond to the two smallest positive roots. These are

$$\boldsymbol{\alpha}_1 \equiv (0, 1)\,, \quad \text{and} \quad \boldsymbol{\alpha}_2 \equiv (1, -1)\,. \tag{40}$$

It is a simple matter to check that the other two positive roots can be expressed as sums of simple roots,

$$(1, 0) = \boldsymbol{\alpha}_1 + \boldsymbol{\alpha}_2\,, \qquad\qquad (1, 1) = 2\boldsymbol{\alpha}_1 + \boldsymbol{\alpha}_2\,.$$

The Cartan matrix is defined by:[5]

$$A_{ij} = \frac{2(\boldsymbol{\alpha}_i, \boldsymbol{\alpha}_j)}{(\boldsymbol{\alpha}_i, \boldsymbol{\alpha}_i)}\,, \tag{41}$$

where the inner product $(\boldsymbol{\alpha}, \boldsymbol{\beta}) \equiv \sum_i \alpha_i \beta_i$ in the convention where $g_{ij} = \delta_{ij}$. Using eq. (40), we obtain $A_{11} = A_{22} = 2$, $A_{12} = -2$ and $A_{21} = -1$. That is,

$$A = \begin{pmatrix} 2 & -2 \\ -1 & 2 \end{pmatrix}\,. \tag{42}$$

The structure of the Dynkin diagram depends on the angle between the two simple roots:

$$\cos\varphi_{\boldsymbol{\alpha}_1\boldsymbol{\alpha}_2} = \frac{(\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2)}{\sqrt{(\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_1)(\boldsymbol{\alpha}_2, \boldsymbol{\alpha}_2)}} = \frac{-1}{\sqrt{2}}\,.$$

Hence $\varphi_{\boldsymbol{\alpha}_1\boldsymbol{\alpha}_2} = 135°$, which corresponds to a double line connecting the two balls of the Dynkin diagram. Hence, the Dynkin diagram corresponding to the Lie algebra, whose simple roots are given by eq. (40), is exhibited in Fig. 2, where the shaded ball corresponds

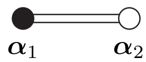


**Figure 2:** The Dynkin diagram for $\mathfrak{sp}(2) \cong \mathfrak{so}(5)$.

to the simple root of the smallest length. In Cartan's notation, this Lie algebra is $B_2 \cong C_2$, which corresponds to $\mathfrak{sp}(2) \cong \mathfrak{so}(5)$ as noted at the end of part (a).

(c) The fundamental weights $\boldsymbol{m}_i$ are defined such that

$$\frac{2(\boldsymbol{m}_i, \boldsymbol{\alpha}_j)}{(\boldsymbol{\alpha}_j, \boldsymbol{\alpha}_j)} = \delta_{ij}\,, \qquad \text{for } i, j = 1, 2, \ldots, r\,, \tag{43}$$

where $\boldsymbol{\alpha}_j \in \Pi$ (the set of simple roots) and $r$ is the rank of the Lie algebra. Using the results of part (b), determine the fundamental weights.

We can solve for the $\boldsymbol{m}_i$ by expanding the fundamental weight vectors in terms of the simple roots:

$$\boldsymbol{m}_i = \sum_{k=1}^{r} c_{ki}\boldsymbol{\alpha}_k\,.$$

Inserting this expression into eq. (43) yields,

$$\sum_{k=1}^{r} c_{ki}\frac{2(\boldsymbol{\alpha}_k, \boldsymbol{\alpha}_j)}{(\boldsymbol{\alpha}_j, \boldsymbol{\alpha}_j)} = \delta_{ij}\,,$$

which can be expressed in terms of the Cartan matrix $A$,

$$\sum_{k=1}^{r} c_{ki}A_{jk} = \delta_{ij}\,.$$

This implies that $c = A^{-1}$, and we conclude that

$$\boldsymbol{m}_i = \sum_{k=1}^{r} (A^{-1})_{ki}\boldsymbol{\alpha}_k\,. \tag{44}$$

Using the Cartan matrix given in eq. (42), the inverse is easily obtained:

$$A^{-1} = \tfrac{1}{2}\begin{pmatrix} 2 & 2 \\ 1 & 2 \end{pmatrix}.$$

Thus, eq. (44) yields the two fundamental weights of $\mathfrak{sp}(2) \cong \mathfrak{so}(5)$,

$$\boldsymbol{m}_1 = \boldsymbol{\alpha}_1 + \tfrac{1}{2}\boldsymbol{\alpha}_2 = (\tfrac{1}{2}\,,\,\tfrac{1}{2})\,, \tag{45}$$

$$\boldsymbol{m}_2 = \boldsymbol{\alpha}_1 + \boldsymbol{\alpha}_2 = (1\,,\,0)\,, \tag{46}$$

where we have used eq. (40) for the simple roots.

[5] *Warning:* in the mathematics literature, eq. (41) is often employed as the definition of the transposed Cartan matrix. You should check carefully when using results from books on Lie algebras.

Each of the two fundamental weights is the highest weight for a particular irreducible representation of $\mathfrak{sp}(2) \cong \mathfrak{so}(5)$, sometimes called a fundamental representation. In general, a simple Lie algebra of rank $\ell$ possesses precisely $\ell$ inequivalent fundamental irreducible representations. Given a highest weight, it is possible to construct all the weights of the corresponding irreducible representation by a technique that is exhibited in Appendix A. Thus, one can construct the weight diagrams corresponding to the two fundamental representations corresponding to $\boldsymbol{m}_1$ and $\boldsymbol{m}_2$, respectively (cf. Fig. 4 in Appendix A).

## Appendix A: Obtaining a complete set of weights

The complete set of weights for the irreducible representations of $\mathfrak{sp}(2) \cong \mathfrak{so}(5)$ corresponding to the highest weights $\boldsymbol{m}_1$ and $\boldsymbol{m}_2$, respectively, can be obtained by the method of block weight diagrams described in Robert N. Cahn, *Semi-Simple Lie Algebras and Their Representations* (Dover Publications, Inc., Mineola, NY, 2006).[6] Given a highest weight $\boldsymbol{M}$, the corresponding *Dynkin labels* $k_i$ (which are non-negative integers) are defined by[7]

$$k_i \equiv \frac{2(\boldsymbol{M}, \boldsymbol{\alpha}_i)}{(\boldsymbol{\alpha}_i, \boldsymbol{\alpha}_i)}\,, \qquad \text{where } \boldsymbol{\alpha}_i \in \Pi\,. \tag{47}$$

The irreducible representations of the Lie algebra $\mathfrak{g}$ are often denoted by placing the $i$th Dynkin label $k_i$ above the $i$th ball of the Dynkin diagram (which corresponds to the $i$th simple root $\boldsymbol{\alpha}_i$). Thus, the block weight diagrams corresponding to the two fundamental representations of $\mathfrak{sp}(2) \cong \mathfrak{so}(5)$ are exhibited below.


**Figure 3:** The block weight diagrams of the fundamental irreducible representations of $\mathfrak{sp}(2) \cong \mathfrak{so}(5)$.

The block weight diagrams are obtained using the theorem quoted in class that establishes strings of weights of the form

$$\frac{2(\boldsymbol{m}, \boldsymbol{\alpha}_i)}{(\boldsymbol{\alpha}_i, \boldsymbol{\alpha}_i)} - kA_{ji}, \qquad \text{for values of } k = 0, 1, 2, , \ldots, \frac{2(\boldsymbol{m}, \boldsymbol{\alpha}_i)}{(\boldsymbol{\alpha}_i, \boldsymbol{\alpha}_i)}\,.$$

[6]However, note that Cahn defines the Cartan matrix that is the transpose of our definition.

[7]Do not confuse the Dynkin labels of a weight with its coordinates in weight space given in eqs. (45) and (46). For example, the fundamental weight $\boldsymbol{m}_1 = \boldsymbol{\alpha}_1 + \frac{1}{2}\boldsymbol{\alpha}_2 = (\frac{1}{2}\,,\,\frac{1}{2})$, whereas its Dynkin labels are $(k_1, k_2) = (1\,,\,0)$.

Thus, starting with any weight $\boldsymbol{m}$, the Dynkin labels for the weights appearing below it in the block weight diagram are obtained by subtracting off the $j$th column of the Cartan matrix $n$ times, where $n$ is the $j$th positive Dynkin label of the weight.[8] Applying the above algorithm has produced the Dynkin labels of the four weights corresponding to the representation specified by $\boldsymbol{m}_1$ and the five weights corresponding to representation specified by $\boldsymbol{m}_2$. In this method, the computation of the multiplicity of a given weight requires additional analysis. But, for the simple cases treated above, all weights appear with multiplicity equal to one, in which case the dimension of the representation is simply equal to the number of weights in the block weight diagram.

Hence, the representations depicted by the block weight diagrams of Fig. 3 are four-dimensional and five-dimensional, respectively, The four-dimensional representation, corresponding to the highest weight $\boldsymbol{m}_1$, is precisely the matrix representation given in eq. (27). This is either the defining representation of $\mathfrak{sp}(2)$ or the spinor representation of $\mathfrak{so}(5)$.[9] In contrast, $\boldsymbol{m}_2$ is the highest weight of a five-dimensional representation, which corresponds to the defining representation of $\mathfrak{so}(5)$.

It is instructive to re-express the weights in terms of its coordinates in the vector space spanned by the simple roots. The weights can then be depicted as vectors in a two-dimensional plane. Given a weight specified by its Dynkin labels $(k_1, k_2)$, the corresponding weight $\boldsymbol{m}$ is obtained by solving the equations [cf. eq. (47)]:

$$k_1 \equiv \frac{2(\boldsymbol{m}, \boldsymbol{\alpha}_1)}{(\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_1)}\,, \qquad\qquad k_2 \equiv \frac{2(\boldsymbol{m}, \boldsymbol{\alpha}_2)}{(\boldsymbol{\alpha}_2, \boldsymbol{\alpha}_2)}\,. \tag{48}$$

To solve for $\boldsymbol{m}$ in terms of $k_1$ and $k_2$, we expand $\boldsymbol{m}$ as a linear combination of simple roots [which are given explicitly in eq. (40)],

$$\boldsymbol{m} = c_1\boldsymbol{\alpha}_1 + c_2\boldsymbol{\alpha}_2\,. \tag{49}$$

Inserting this expression for $\boldsymbol{m}$ into eq. (48), it follows that:

$$\begin{aligned} k_1 &= \frac{2(c_1\boldsymbol{\alpha}_1 + c_2\boldsymbol{\alpha}_2\,,\, \boldsymbol{\alpha}_1)}{(\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_1)} = 2c_1 - 2c_2\,, \\ k_2 &= \frac{2(c_1\boldsymbol{\alpha}_1 + c_2\boldsymbol{\alpha}_2\,,\, \boldsymbol{\alpha}_2)}{(\boldsymbol{\alpha}_2, \boldsymbol{\alpha}_2)} = -c_1 + 2c_2\,, \end{aligned}$$

where we have used $(\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_1) = 1$, $(\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2) = -1$ and $(\boldsymbol{\alpha}_2, \boldsymbol{\alpha}_2) = 2$. Solving for $c_1$ and $c_2$ then yields:

$$c_1 = k_1 + k_2\,, \qquad\qquad c_2 = \tfrac{1}{2}k_1 + k_2\,. \tag{50}$$

Hence, using eqs. (40) and (50), the weight $\boldsymbol{m}$ specified by eq. (49) is given by:

$$\boldsymbol{m} = \left(\tfrac{1}{2}k_1 + k_2\,,\, \tfrac{1}{2}k_1\right)\,. \tag{51}$$

[8]If there is more than one positive Dynkin label, then the block weight diagram branches. This does not occur in the present example.

[9]Since $\mathfrak{sp}(2) \cong \mathfrak{so}(5)$, one is free to regard the representations obtained above as representations of either Lie algebra. However, the interpretation of the representation depends on which choice of Lie algebra is made.

As a check, if $\boldsymbol{m} = \boldsymbol{m}_1$ then $k_1 = 1$ and $k_2 = 0$, in which case $c_1 = 1$, $c_2 = \frac{1}{2}$ and $\boldsymbol{m}_1 = (\frac{1}{2}\,,\,\frac{1}{2})$ in agreement with eq. (45). Likewise, if $\boldsymbol{m} = \boldsymbol{m}_2$ then $k_1 = 0$ and $k_2 = 1$, in which case $c_1 = c_2 = 1$ and $\boldsymbol{m}_1 = (1\,,\,0)$ in agreement with eq. (46).

One can use eq. (51) to obtain the coordinates of all the weights exhibited in Fig. 3. For the four-dimensional representation specified by the Dynkin labels $(1, 0)$ and the five-dimensional representation specified by the Dynkin labels $(0, 1)$, the corresponding weight space diagrams are given in Fig. 4.[10]

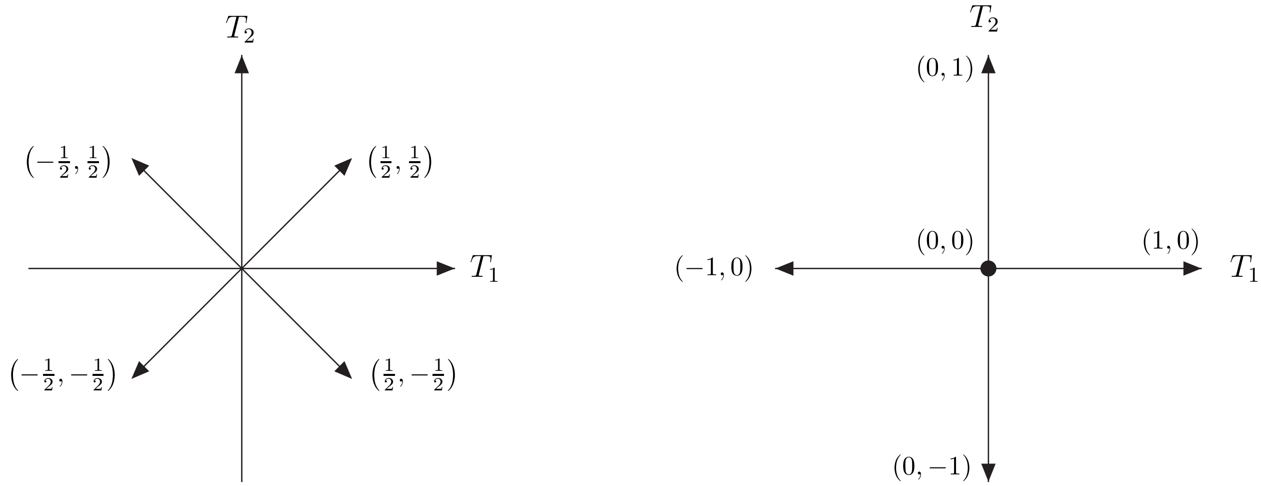


**Figure 4:** The weight diagrams of the irreducible representations of $\mathfrak{sp}(2) \cong \mathfrak{so}(5)$, with dimensions four [left] and five [right], respectively.

In particular, $T_1 \equiv \frac{1}{2}H_1 = \frac{1}{2}C_3$ and $T_2 \equiv \frac{1}{2}H_2 = \frac{1}{2}B_3$ are the diagonal generators normalized such that the shortest root vector has length 1. Given the explicit four-dimensional representation in eq. (27), one can check that the weight vectors exhibited in Fig. 4, $\bigl\{(\frac{1}{2},\frac{1}{2})\,,\,(\frac{1}{2},-\frac{1}{2})\,,\,(-\frac{1}{2},\frac{1}{2}),(-\frac{1}{2},-\frac{1}{2})\bigr\}$,[11] satisfy the eigenvalue equations,[12]

$$T_i|\boldsymbol{m}\rangle = m_i|\boldsymbol{m}\rangle\,, \qquad \text{for } i = 1, 2\,, \tag{52}$$

where $\boldsymbol{m} = (m_1, m_2)$ are the coordinates in the $T_1$–$T_2$ plane. The weights of the five-dimensional representation shown in Fig. 4, $\bigl\{(1,0),(0,1),(0,0),(0,-1),(-1,0)\bigr\}$,[13] include a zero weight (indicated by the filled circle at the origin of the weight diagram). To check that eq. (52) is satisfied in this latter case, it is straightforward to construct explicit five-dimensional matrix representations of $T_1$ and $T_2$, which are the diagonal generators of the defining representation of $\mathfrak{so}(5)$.

[10]As previously noted, all weights shown in the two weight space diagrams above have multiplicity one, which means that the corresponding simultaneous eigenvector $|\boldsymbol{m}\rangle$ defined in eq. (52) is unique.

[11]The corresponding Dynkin indices, obtained in Fig. 3, are $(k_1, k_2) = (1, 0)\,,\,(-1, 1)\,,\,(1, -1)$ and $(-1, 0)$, respectively.

[12]Sometimes, the eigenvalues $m_1$ and $m_2$ are called *weights* and the corresponding eigenvector $|\boldsymbol{m}\rangle$ is called the weight vector. However, it is more common to refer to the *weight vector* $\boldsymbol{m}$ of a weight space diagram as the vector whose coordinates $(m_1, m_2)$ are given by the eigenvalues of $T_1$ and $T_2$.

[13]The corresponding Dynkin indices, obtained in Fig. 3, are $(k_1, k_2) = (0, 1)\,,\,(2, -1)\,,\,(0, 0)\,,\,(-2, 1)$ and $(0, -1)$, respectively.