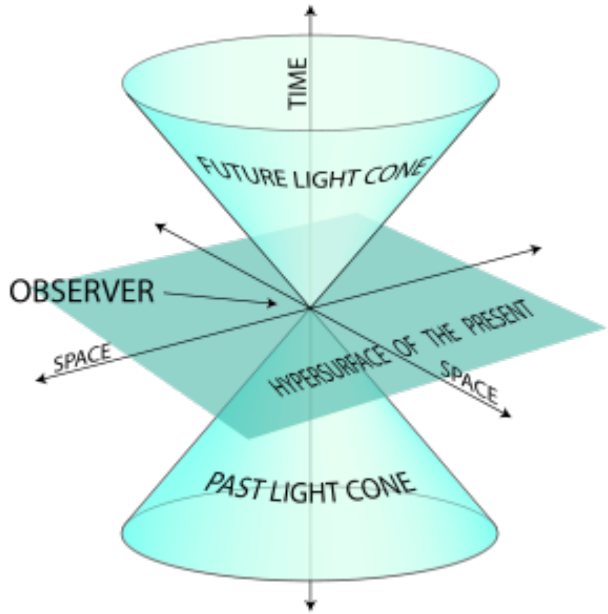
The Lorentz and Poincaré groups

By Joel Oredsson



The Principle of Special Relativity:

The laws of nature should be covariant with respect to the transformations between inertial reference frames.

$$x^{\mu} \rightarrow x'^{\mu} = f^{\mu}(x)$$

$$g_{\mu\nu} x^{\mu} x^{\nu} = g_{\rho\sigma} x'^{\rho} x'^{\sigma} = x_0{}^2 - \vec{x}^2$$

$$\Rightarrow g_{\mu\upsilon} = g_{\rho\sigma} \frac{\partial x'^{\rho}}{\partial x^{\mu}} \frac{\partial x'^{\sigma}}{\partial x^{\upsilon}} = g_{\rho\sigma} \frac{\partial f^{\rho}}{\partial x^{\mu}} \frac{\partial f^{\sigma}}{\partial x^{\upsilon}}$$

We find that the transformation f(x) is linear and the transformation matrix has det=±1.

$$\frac{\partial^2 f^\rho}{\partial x^\mu \partial x^\nu} = 0 \qquad \det\left(\frac{\partial f^\rho}{\partial x^\mu}\right) = \pm 1$$

$$\Rightarrow x'^\mu = \Lambda^\mu_\nu x^\nu + a^\mu = g(\Lambda, a)x$$

Group multipication law

$$g(\Lambda', a')g(\Lambda, a) = g(\Lambda'\Lambda, \Lambda' a + a')$$

We can now define two types of transformations.

Poincaré transformations:

- Translations
- Lorentz Transformations

$$x'^{\mu} = \Lambda^{\mu}_{\nu} x^{\nu} + a^{\mu}$$

Lorentz transformations:
Or homogeneous Lorentz/
Poincaré transformations.

- Rotations
- Boosts

$$x'^{\mu} = \Lambda^{\mu}_{\nu} x^{\nu}$$

The Lorentz group:

The group of all Lorentz transformations, restricted by

$$g_{\mu\nu}\Lambda^{\mu}{}_{\rho}\Lambda^{\nu}{}_{\sigma} = g_{\rho\sigma}$$

$$\Rightarrow (\Lambda^{0}_{0})^{2} = 1 + \sum_{i} \Lambda^{i}_{0}\Lambda^{i}_{0} \geq 1 \qquad \det(\Lambda) = \pm 1$$

The full Lorentz group consists of 4 disconnected pieces.

We can decompose the Lorentz group as a cosets of the proper orthocronous(restricted) Lorentz Group.

$$L_{+}^{\uparrow} : \det(\Lambda) = 1, \Lambda_{0}^{0} \geq 1$$

$$L = L_{+}^{\uparrow} \bigcup PL_{+}^{\uparrow} \bigcup TL_{+}^{\uparrow} \bigcup PTL_{+}^{\uparrow}$$

	Orthocronous $\Lambda_{0}^{0} \geq 1$	Antichronous,T $\Lambda_{0}^{0} \leq -1$
Proper $\det \Lambda = 1$	No reversals	Time reversals
Improper,P $\det \Lambda = -1$	Space inversion	Space and time inversions.

The group name for the restricted Lorentz group is SO(1,3). It can be represented by a 4x4 matrix.

$$g_{\rho\sigma} = g_{\mu\nu}\Lambda^{\mu}{}_{\rho}\Lambda^{\nu}{}_{\sigma} = g_{\mu\nu}(\delta^{\mu}_{\rho} + \omega^{\mu}_{\rho})(\delta^{\nu}_{\sigma} + \omega^{\nu}_{\sigma}) + \mathrm{O}(\omega^2)$$

$$= g_{\rho\sigma} + \omega_{\sigma\rho} + \omega_{\rho\sigma} + \mathrm{O}(\omega^2)$$

$$\Rightarrow \omega_{\sigma\rho} = -\omega_{\rho\sigma}$$

The parameter matrix is antisymmetric with 6 independent variables. 3 for boosts, 3 for rotations.

The generators of SO(1,3) explicitly:

$$K_1 = \begin{pmatrix} 0 & i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, K_2 = \begin{pmatrix} 0 & 0 & i & 0 \\ 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, K_3 = \begin{pmatrix} 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix}$$

$$J_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{pmatrix}, J_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{pmatrix}, J_3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

A general finite Lorentz transformation can then be written as

$$\Lambda(\omega) = \exp[-\frac{i}{2}\omega^{\mu\nu} S_{\mu\nu}]$$

Where

$$J_i = \frac{1}{2}\varepsilon^{ijk} S_{jk} \qquad K_i = S_{i0}$$

And the Lie algebra for the Lorentz group is

$$[J_i, J_j] = i\varepsilon^{ijk} J_l$$

$$[K_i, J_j] = i\varepsilon^{ijk} K_l$$

$$[K_i, K_j] = -i\varepsilon^{ijk} J_l$$

If we compexify SO(1,3) to SO(4,C) we find something interesting

$$M_i = \frac{1}{2}(J_i + iK_i) \qquad N_i = \frac{1}{2}(J_i - iK_i)$$

$$\Rightarrow \begin{cases} [M_i, M_j] = i\varepsilon^{ijk} M_l \\ [N_i, N_j] = i\varepsilon^{ijk} N_l \\ [M_i, N_j] = 0 \end{cases} \qquad so(4,C) \cong sl(2,C) \oplus sl(2,C)$$

We know the representations of sl(2,C) so we can use these to find a represention of SO(1,3). These will be labeled by the highest weight (j,j') where each ranging from 0, ½, 1, 3/2 , 2, … . Each represantation being (2j+1)(2j'+1) dimensional.

The simplest representations:

dim		
1	(0,0)	Scalar
2	(½,0)	Left handed Weyl spinor
2	(0,½)	Right handed Weyl spinor
4	(½,½)	4 vector

For example the Weyl spinors are 2 dimensional objects transforming as

$$\psi_L \to \Lambda\psi_L = \exp\left[(-i\vec{\theta} - \vec{\eta})\cdot \frac{\vec{\sigma}}{2}\right]\psi_L$$

$$\psi_R \to \Lambda\psi_R = \exp\left[(-i\vec{\theta} + \vec{\eta})\cdot \frac{\vec{\sigma}}{2}\right]\psi_R$$

With these we can create the reducible Dirac spinor (4dim)

$$\psi_D = \begin{pmatrix}\psi_L \\ \psi_R\end{pmatrix} = \left(\frac{1}{2},0\right) \oplus \left(0,\frac{1}{2}\right)$$

Relationship to SL(2,C)

Consider the mapping $f : x^{\mu} \rightarrow X = \sigma_{\mu} x^{\mu} = \begin{pmatrix} x^0 - x^3 & -x^1 + ix^2 \\ -x^1 - ix^2 & x^0 + x^3 \end{pmatrix}$

A linear transformation on X preserves its determinant which corresponds to the length of x.

$$\det(X) = (x^0)^2 - \vec{x}^{\,2}$$

A Lorentz transformation on x can then be mapped into a transformation on X.

$$x' = \Lambda x \underset{Lorentz \quad transformation}{\Leftrightarrow} X' = AXA^{\dagger}$$

If we fix detA=1, then every A belongs to SL(2,C). The transformation of A preserves hermicity of X.

$$X' = x'^{\mu} \sigma_{\mu} = \Lambda(A)^{\mu}_{\nu} x^{\nu} \sigma_{\mu}$$

To find the matrices we can consider infinitesimal transformations to find the generators of the transformation in the SL(2,C) representation. For example consider a rotation about the z-axis.

$$x'^1 = x^1 - \delta\theta x^2, x'^2 = x^2 + \delta x^1$$

$$A = I - i\delta\theta J_3$$

$$X = \sigma_\mu x^\mu + \delta\theta(-\sigma_1 x^2 + \sigma_2 x^1) = AXA^\dagger = X - i\delta\theta(J_3 X - XJ_3{}^\dagger)$$

Compare both sides to find

$$J_3 = \frac{\sigma_3}{2}$$

A rotation by an angle theta around an axis n corresponds to in SL(2,C) the matrix(SU(2))

$$A = \pm \exp[-i\frac{\theta}{2}\vec{n}\cdot\vec{\sigma}]$$

Similarly, a boost in the n direction by rapidity xi can be expressed by

$$A = \pm \exp[-\frac{\xi}{2}\vec{n}\cdot\vec{\sigma}]$$

So we do have a homomorphism between SO(1,3) and SL(2,C). Noticing the ± sign we see that the mapping is 1 to 2 and similarly to SU(2) and SO(3) we have

$$SO^{+}(1,3) \cong SL(2,C)/Z_2$$

Generators of the Poincaré group

An element of the Poincaré group can be expressed as a 5x5 matrix g. It is a 10 parameter group.
6 for a Lorentz transformation
4 for translations

$$g(\vec{a},\Lambda) = \begin{pmatrix} & & & & a^0 \\ & & & & a^1 \\ & & \Lambda & & a^2 \\ & & & & a^3 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} = T(a)g(0,\Lambda)$$

The Lie algebra is given by

$$[P_\mu, P_\nu] = 0$$

$$[P_\mu, J_{\lambda\sigma}] = i(P_\lambda g_{\mu\sigma} - P_\sigma g_{\mu\lambda})$$

$$[J_{\mu\nu}, J_{\lambda\sigma}] = i(J_{\lambda\nu} g_{\mu\sigma} - J_{\sigma\nu} g_{\mu\lambda} + J_{\mu\lambda} g_{\nu\sigma} - J_{\mu\sigma} g_{\nu\lambda})$$

Translations is an invariant subgroup.
The Poincaré group is a semidirect product of translations and Lorentz transformations.

Transformations in quantum theories

In a Hilbert space the symmetry should manifest itself in the form of unitary operators.

Lorentz Group is non compact -> no finite dimensional unitary irreps

$$|\psi\rangle \rightarrow U(\Lambda,a)|\psi\rangle$$

$$U(\Lambda,a) = U_T(a)U_\Lambda(\Lambda)$$

$$U(\Lambda',a')U(\Lambda,a) = U(\Lambda'\Lambda,\Lambda' a + a')$$

One particle states

In a unitary transform the generators are hermitian. We can express the physical state vectors as eigenvectors to the energy-momentum operator.

$$P^{\mu}\left|p,\sigma\right\rangle = p^{\mu}\left|p,\sigma\right\rangle \Rightarrow e^{-ib_{\mu}P^{\mu}}\left|p,\sigma\right\rangle = e^{-ib_{\mu}p^{\mu}}\left|p,\sigma\right\rangle$$

The state vectors are then labeled by the 4 momenta and sigma: all remaining degrees of freedom.

We know how P transforms during a Lorentz Transformation.

$$U^{-1}(\Lambda)P^{\mu}U(\Lambda) = \Lambda^{\mu}_{\nu}P^{\mu}$$

$$P^{\mu}U(\Lambda)\left|p,\sigma\right\rangle = U(\Lambda)\left[U^{-1}(\Lambda)P^{\mu}U(\Lambda)\right]\left|p,\sigma\right\rangle = \Lambda^{\mu}_{\nu}p^{\nu}U(\Lambda)\left|p,\sigma\right\rangle$$

$$\Rightarrow U(\Lambda)\left|p,\sigma\right\rangle = \sum_{\sigma'} C_{\sigma'\sigma}(\Lambda,p)\left|\Lambda p,\sigma'\right\rangle$$

Method of induced representations

- Choose a standard 4 momentum vector
- Identify its Little group
- Find the irreducible representations of the Little group.
- For each of these, apply Lorentz transformations to get the full representation.

So for every W that

$$W^{\mu}{}_{\nu} k^{\nu} = k^{\mu}$$

We have

$$U(W)|k,\sigma\rangle = \sum_{\sigma'} D_{\sigma'\sigma}(W)|k,\sigma'\rangle$$

$$U(\Lambda)|p,\sigma\rangle = \sum_{\sigma'} D_{\sigma'\sigma}\big(W(\Lambda,p)\big)|\Lambda p,\sigma'\rangle$$

$$W(\Lambda,p) = L^{-1}(\Lambda p)\Lambda L(p) \qquad k \underset{L(p)}{\longrightarrow} p \underset{\Lambda}{\longrightarrow} \Lambda p \underset{L^{-1}(\Lambda p)}{\longrightarrow} k$$

We can label the standard momentum vectors with the eigenvalue of the Casimir operators.

$$P^{\mu} P_{\mu} \qquad\qquad W^{\mu} W_{\mu}$$

Where W is the Pauli-Lubanski vector. $W^{\mu} = \frac{1}{2}\varepsilon^{\mu\nu\rho\sigma} J_{\nu\rho} P_{\sigma}$

	p^2, p^0	p^μ	$Little \quad Group$
$a) Time-like$	$p^2 = m^2 > 0, p^0 > 0$	$(m,0,0,0)$	$SO(3)$
$b) Time-like$	$p^2 = m^2 > 0, p^0 < 0$	$(-m,0,0,0)$	$SO(3)$
$c) Light-like$	$p^2 = 0, p^0 > 0$	$(\omega,0,0,\omega)$	$E(2)$
$d) Light-like$	$p^2 = 0, p^0 < 0$	$(-\omega,0,0,\omega)$	$E(2)$
$e) Space-like$	$p^2 = -n^2 < 0$	$(0,0,0,n)$	$SO(2,1)$
$f) null-vector$	$p^\mu = 0$	$(0,0,0,0)$	$SO(3,1)$

So for the case a we have a massive particle labeled by mass, intrinsic spin, momentum and helicity.

$$\left| m, s, \vec{p}, \sigma \right\rangle$$

Massive

Familiar SU(2) representations from ordinary QM.

s=0,1/2,1,3/2… (2s+1) degrees of freedom

$$T(b)\left|p,\sigma\right\rangle = e^{-ib_\mu p^\mu}\left|p,\sigma\right\rangle$$

$$\Lambda\left|p,\sigma\right\rangle = \sum_{\sigma'}\left|\Lambda p,\sigma'\right\rangle D^s{}_{\sigma'\sigma}\left(R(\Lambda,p)\right)$$

Massless

Has quantized helicity.

s=0,±1/2,±1,… 1 degree of freedom

The familiar photon is a mix of two states with helicities ±1.

$$T(b)\left|p,\sigma\right\rangle = e^{-ib_\mu p^\mu}\left|p,\sigma\right\rangle$$

$$\Lambda\left|p,\sigma\right\rangle = \left|\Lambda p,\sigma\right\rangle e^{-i\sigma\theta(\Lambda,p)}$$

What we have found:

- The Poincaré group is a 10 dimensional non-compact Lie group with the Lorentz group as a subgroup.
- The Lorentz group can be divided into 4 cosets of the proper Lorentz group.It is a doubly connected group with SL(2,C) as the universal covering group. SO(1,3)≈SL(2,C)/Z_2
- The irreducible finite representations (j,j’) can be used to construct fields that have well defined transformation rules under Poincare transformations.
- The infinte irreps are used to characterize all possible particle states. Massive particles is characterized by spin j and have (2j+1) degrees of freedom. Massless particles are labeled by helicity ±integer/half-integer.

References:

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