# Group theory of monopoles

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Dirac quantization condition:

$$qg = 2\pi n, \quad n \in \mathbb{Z} \tag{1}$$

$q$: electric charge

$g$: magnetic charge

Existence of magnetic monopoles would imply charge quantization

# Dirac Monopoles

Suppose we have

$$\mathbf{B} = \frac{g}{4\pi r^2}\hat{r}.$$

The associated vector potential is singular, not only at the origin, but along a "Dirac string" extending from the origin to infinity. Requiring that the Dirac string not be physically observable leads to the quantization condition.

# Group theoretic approach

Consider a theory with a gauge group $G$ spontaneously broken down to a subgroup $H$ by a Higgs field $\Phi$. Let $\mathcal{M}_0$ be the vacuum manifold of $\Phi$:

$$\mathcal{M}_0 = \{\Phi | V(\Phi) = V_{\min}\} . \quad (2)$$

We can choose a particular point $\Phi_0$ in $\mathcal{M}_0$, and then

$$\mathcal{M}_0 = \{g\Phi_0 | g \in G\} = \mathrm{orbit}_G(\Phi_0) \quad (3)$$

The little group of $\Phi_0$ with respect to $G$ is

$$H = \{g \in G | g\Phi_0 = \Phi_0\} . \quad (4)$$

Since

$$\mathrm{orbit}_G(\Phi_0) \cong G/H, \quad (5)$$

we have

$$\mathcal{M}_0 \cong G/H. \quad (6)$$

In the Higgs vacuum, the only non-zero component of the gauge field tensor is $F^{\mu\nu}$, which satisfies Maxwell's equations (no magnetic monopoles).

But $\Phi$ need not satisfy the vacuum condition everywhere. This can give rise to monopoles.

# Magnetic flux through a surface

Consider the magnetic flux, call it $g_\Sigma$, through some closed surface $\Sigma$ on which the Higgs potential is minimized:

$$g_\Sigma = \int_\Sigma \mathbf{B} \cdot d\mathbf{\Sigma}. \tag{7}$$

Turns out this depends only on the values of $\Phi$ on the surface $\Sigma$. And in fact, continuous deformations of $\Phi$ do not affect $g_\Sigma$. So actually $g_\Sigma$ depends only on the homotopy classes of the maps

$$\Phi : \Sigma \to \mathcal{M}_0$$

# Quantization

To have finite energies, $\Phi(\vec{r})$ must approach a point on $\mathcal{M}_0$ as $r$ goes to infinity:

$$\lim_{r\to\infty} \Phi(\vec{r}) \equiv \Phi_\infty(\hat{r}) \in \mathcal{M}_0 \tag{8}$$

where $\vec{r} = r\hat{r}$ is a position vector in spherical coordinates.

In $d+1$ dimensions, $\hat{r} \in S^{d-1}$ (e.g. in 3+1 dimensions, $\hat{r} \in S^2$).

Take the surface surface at infinity ($S^{d-1}$, where $\Phi$ satisfies the vacuum condition) as the surface $\Sigma$.Then the flux $g_{S^{d-1}}$ depends on the homotopy classes of the maps

$$\Phi_\infty : S^{d-1} \to \mathcal{M}_0. \tag{9}$$

We call the group of these homotopy classes $\pi_{d-1}(\mathcal{M}_0)$. Since $\mathcal{M}_0 \cong G/H$, we are interested in the group

$$\pi_{d-1}(G/H)$$

If (and only if) all field configurations of finite energy are homotopically equivalent (i.e. may be continuously deformed into each other), then we have $\pi_{d-1}(G/H) = \{e\}$. If, however, $\pi_{d-1}(G/H) \neq \{e\}$, then we have so-called "topological solitons," or monopoles. That is, we have field configurations of finite energy which cannot be continuously deformed into one another.

# 't Hooft-Polyakov monopoles

Gauge group $G = SO(3)$, broken by Higgs v.e.v. $\phi_0 = (0, 0, v)$

Remaining symmetry group is

$$H = \text{rotations about the } \phi \text{ axis} = SO(2) \cong U(1)$$

Generators of $G$ are $T^a$ for $a = 1, 2, 3$.

Generator of $H$ is $\phi^a T^a / v$

Associate the $U(1)$ symmetry with electromagnetism. Electric charge is then $Q = \frac{e\hbar}{v} \phi^a T^a$ (and the $T^a$ have half-integer eigenvalues).

# 't Hooft-Polyakov, continued

Then $\mathcal{M}_0 \cong G/H = SO(3)/SO(2)$ is isomorphic to the two-sphere $S^2$. In $3+1$ dimensions,

$$\pi_{d-1}(G/H) = \pi_2\left(S^2\right) = \mathbb{Z}.$$

We can think of the equivalence classes as being characterized by the number of times, $N$, that a two-dimensional surface is wrapped around the sphere $\mathcal{M}_0$.

This number $N$ completely determines the homotopy class.

# How is this related to quantization of charge?

If $G$ is simply connected, then $\pi_2(G/H) \cong \pi_1(H)$. $SO(3)$ is not simply connected, but we can replace it like $SU(2)$ to proceed.

In this case we just need to consider closed paths in $H$. For the case of $SO(3)$ broken to $U(1)$, these have the form

$$h(s) = \exp\left(iq \int_{\Sigma} \mathbf{B} \cdot d\mathbf{\Sigma}\right) = e^{iqg}, \qquad 0 \leq s \leq 1, \tag{10}$$

and the requirement $h(0) = h(1)$ leads to the Dirac quantization condition:

$$qg = 2\pi N, \qquad N \in \mathbb{Z}. \tag{11}$$

# Conceptualizing topological solitons

As an illustration of this idea, consider the Sine-Gordon model, which can be thought of as a long clothesline of identical pegs, connected to each other by identical springs, and acted on by gravity.

Ground state: they all hang straight down.

A stable state of finite non-zero energy: the pegs are twisted by an integer multiple of $2\pi$, but the pegs at infinity in either direction hang straight down. Would take infinite energy to flip the pegs at infinity, so this state never decays. This is called a "kink" or "soliton".

# References


- P. Goddard and D. I. Olive. *Magnetic monopoles in gauge field theories* Rep. Prog. Phys., Vol. 41, 1978
- F. Alexander Bais. *To be or not to be? Magnetic monopoles in non-abelian gauge theories*. hep/th/0407197, 2004
- H. Haber. UCSC Physics 251 Lecture notes and class handouts, Spring 2013