Let A be a complex $d \times d$ antisymmetric matrix, i.e. $A^{\mathsf{T}} = -A$. Since

$$\det A = \det\,(-A^{\mathsf{T}}) = \det\,(-A) = (-1)^d \det A\,, \tag{1}$$

it follows that $\det A = 0$ if d is odd. Thus, the rank of A must be even. In these notes, the rank of A will be denoted by $2n$. If $d \equiv 2n$ then $\det A \neq 0$, whereas if $d > 2n$, then $\det A = 0$. All the results contained in these notes also apply to real antisymmetric matrices.

Theorem 1: If A is an even-dimensional complex invertible $2n \times 2n$ antisymmetric matrix, then there exists an invertible $2n \times 2n$ matrix P such that:

$$A = P^{\mathsf{T}} J P\,, \tag{2}$$

where the $2n \times 2n$ matrix J written in 2×2 block form is given by:

$$J \equiv \operatorname{diag}\underbrace{\left\{\begin{pmatrix} 0 & 1 \\ -1 & 0\end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0\end{pmatrix}, \cdots, \begin{pmatrix} 0 & 1 \\ -1 & 0\end{pmatrix}\right\}}_{n}. \tag{3}$$

If A is a complex singular antisymmetric $d \times d$ matrix of rank $2n$ (where d is either even or odd and $d > 2n$), then there exists an invertible $d \times d$ matrix P such that

$$A = P^{\mathsf{T}} \widetilde{J} P\,, \tag{4}$$

and $\widetilde{J}$ is the $d \times d$ matrix that is given in block form by

$$\widetilde{J} \equiv \left(\begin{array}{c:c} J & \mathbb{O} \\ \hdashline \mathbb{O} & \mathbb{O}\end{array}\right), \tag{5}$$

where the $2n \times 2n$ matrix J is defined in eq. (3) and $\mathbb{O}$ is a zero matrix of the appropriate number of rows and columns.

Proof: Details of the proof of this theorem are given in Appendix B.

For any even-dimensional complex $2r \times 2r$ antisymmetric matrix A, we define the *pfaffian* of A, denoted by $\operatorname{pf} A$, by:

$$\operatorname{pf} A = \frac{1}{2^r r!} \sum_{p \in S_{2r}} (-1)^p\, A_{i_1 i_2} A_{i_3 i_4} \cdots A_{i_{2r-1} i_{2r}}\,, \tag{6}$$

where the sum is taken over all permutations

$$p = \begin{pmatrix} 1 & 2 & \cdots & 2r \\ i_1 & i_2 & \cdots & i_{2r}\end{pmatrix}$$

and $(-1)^p$ is the sign of the permutation $p \in S_{2r}$. If A is an odd-dimensional complex antisymmetric matrix, the corresponding pfaffian is defined to be zero. The following theorem relates the pfaffian and determinant of an antisymmetric matrix.

Theorem 2: If A is a complex antisymmetric matrix, then

$$\det A = [\mathrm{pf}\, A]^2 \,.$$

Proof: First, we assume that A is a complex invertible $2n\times 2n$ antisymmetric matrix. For $n = 1$, $\mathrm{pf}\, A = A_{12}$ and $\det A = (A_{12})^2$, so the theorem clearly holds. Hence, assume that $n > 1$. Using the results of Theorem 1, it follows that

$$\det A = \det(P^{\mathsf{T}} J P) = [\det P]^2 \det J = [\det P]^2 \,, \tag{7}$$

since eq. (3) implies that $\det J = 1$. To compute the pfaffian of A, we use eq. (2) to write

$$\begin{aligned} A_{ij} &= \sum_{k=1}^{2n}\sum_{\ell=1}^{2n} P^k{}_i J_{k\ell} P^\ell{}_j \qquad (i,j=1,2,\ldots,2n) \\ &= P^1{}_i P^2{}_j - P^2{}_i P^1{}_j + P^3{}_i P^4{}_j - P^4{}_i P^3{}_j + \cdots + P^{2n-1}{}_i P^{2n}{}_j - P^{2n}{}_i P^{2n-1}{}_j \,, \end{aligned} \tag{8}$$

We now substitute eq. (8) into the definition of the pfaffian [eq. (6) with $r = n$]. Examine the possible cross-terms that appear in the sum given by eq. (6). Let $j_1, j_2, \ldots, j_n$ be n odd integers (not necessarily distinct) taken from the set $\{1, 3, \ldots, 2n-1\}$. As an example, choose $j_1 = j_2 = k$ for some fixed odd integer k. Then,

$$\begin{aligned} &\frac{1}{2^n}\sum_{p\in S_{2n}} (-1)^p \, (P^k{}_{i_1} P^{k+1}{}_{i_2} - P^k{}_{i_2} P^{k+1}{}_{i_1})(P^k{}_{i_3} P^{k+1}{}_{i_4} - P^k{}_{i_4} P^{k+1}{}_{i_3}) \\ &\qquad\qquad \times (P^{j_3}{}_{i_5} P^{j_3+1}{}_{i_6} - P^{j_3}{}_{i_6} P^{j_3+1}{}_{i_5}) \cdots (P^{j_n}{}_{i_{2n-1}} P^{j_n+1}{}_{i_{2n}} - P^{j_n}{}_{i_{2n}} P^{j_n+1}{}_{i_{2n-1}}) \\ &= \sum_{i_1,i_2,\ldots,i_{2n}} P^k{}_{i_1} P^{k+1}{}_{i_2} P^k{}_{i_3} P^{k+1}{}_{i_4} P^{j_3}{}_{i_5} P^{j_3+1}{}_{i_6} \cdots P^{j_n}{}_{i_{2n-1}} P^{j_n+1}{}_{i_{2n}} \epsilon^{i_1 i_2 \cdots i_{2n}} = 0 \,, \end{aligned} \tag{9}$$

since interchanging either $i_1 \leftrightarrow i_3$ or $i_2 \leftrightarrow i_4$ yields a term of the opposite sign in the sum. Hence, there are an equal number of positive and negative terms that cancel in pairs, and the sum in eq. (9) vanishes as indicated above. It follows that:

$$\begin{aligned} \mathrm{pf}\, A &= \frac{1}{2^n n!}\sum_{p\in S_{2n}} (-1)^p \, (P^k{}_{i_1} P^{k+1}{}_{i_2} - P^k{}_{i_2} P^{k+1}{}_{i_1})(P^k{}_{i_3} P^{k+1}{}_{i_4} - P^k{}_{i_4} P^{k+1}{}_{i_3}) \\ &\qquad\qquad \times \cdots (P^k{}_{i_{2n-1}} P^{k+1}{}_{i_{2n}} - P^k{}_{i_{2n}} P^{k+1}{}_{i_{2n-1}}) \\ &= \sum_{i_1,i_2,\ldots,i_{2n}} P^1{}_{i_1} P^2{}_{i_2} P^3{}_{i_3} P^4{}_{i_4} \cdots P^{2n-1}{}_{i_{2n-1}} P^{2n}{}_{i_{2n}} \epsilon^{i_1 i_2 \cdots i_{2n}} \\ &= \det P \,, \end{aligned} \tag{10}$$

where the factor of $[n!]^{-1}$ is canceled out precisely by the $n!$ identical cross-terms in which all superscript indices of the P's are distinct. The final step of eq. (10) follows from the definition of the determinant. Inserting the result of eq. (10) into eq. (7) then yields

$$\det A = [\operatorname{pf} A]^2 \,, \tag{11}$$

which completes the proof for the case of invertible A. If A is a $2r \times 2r$ singular antisymmetric matrix of rank $2n$ (where $r > n$), then eq. (8) is replaced by:

$$\begin{aligned} A_{ij} &= \sum_{k=1}^{2r}\sum_{\ell=1}^{2r} P^k{}_i \widetilde{J}_{k\ell} P^\ell{}_j = \sum_{k=1}^{2n}\sum_{\ell=1}^{2n} P^k{}_i J_{k\ell} P^\ell{}_j \qquad (i,j = 1,2,\ldots,2r\,, \quad \text{where } r > n) \\ &= P^1{}_i P^2{}_j - P^2{}_i P^1{}_j + P^3{}_i P^4{}_j - P^4{}_i P^3{}_j + \cdots + P^{2n-1}{}_i P^{2n}{}_j - P^{2n}{}_i P^{2n-1}{}_j \,, \end{aligned} \tag{12}$$

where we have used eq. (5) to express $\widetilde{J}$ in terms of J. In this case, when we substitute eq. (12) into eq. (6) we see that some of the products $P^k P^{k+1}$ will necessarily be repeated in each term in the sum because $r > n$. Hence, each resulting term is of a form similar to the one given by eq. (9) and therefore vanishes, and we conclude that $\operatorname{pf} A = 0$. Since A is singular, $\det A = 0$, so eq. (11) is also satisfied in this case.

Finally, if A is an odd-dimensional (complex) antisymmetric matrix, then $\operatorname{pf} A = 0$ by definition and $\det A = 0$ as a result of eq. (1). Hence again eq. (11) is satisfied. Theorem 2 is now proven for any complex antisymmetric matrix.

It is possible to rewrite the above proof using the formalism of exterior products. The derivation of Theorem 2 in this language can be found in *Linear algebra via exterior products*, by Sergei Winitzki (published by `lulu.com`, 2010).[1] The proof that I have provided above is simply a "translation" of Winitzki's proof of Theorem 2 into a language that is more familiar to physics graduate students.

APPENDIX A: Bilinear forms

In this appendix, we introduce the concept of bilinear forms[2] that will be useful for proving Theorem 1 in Appendix B.

Let V be a vector space over the field F. In what follows, we shall assume that $F = \mathbb{C}$ (although the results of this appendix also apply if $F = \mathbb{R}$). A *bilinear form* on V is a function f that assigns to each ordered pair of vectors $\boldsymbol{v}$, $\boldsymbol{w} \in V$ a scalar $f(\boldsymbol{v},\boldsymbol{w}) \in F$ such that

$$\begin{aligned} f(c\boldsymbol{v_1} + \boldsymbol{v_2}, \boldsymbol{w}) &= cf(\boldsymbol{v_1},\boldsymbol{w}) + f(\boldsymbol{v_2},\boldsymbol{w})\,, \\ f(\boldsymbol{v}, c\boldsymbol{w_1} + \boldsymbol{w_2}) &= cf(\boldsymbol{v},\boldsymbol{w_1}) + f(\boldsymbol{v},\boldsymbol{w_2})\,, \end{aligned}$$

[1] See `http://sites.google.com/site/winitzki/linalg#TOC-Get-the-book` for a free copy of this book.

[2] See, e.g., Kenneth Hoffman and Ray Kunze, *Linear Algebra* (Prentice Hall, Inc., Englewood Cliffs, N.J., 1961), chapter 9.

for any constant scalar $c \in F$.

Assume that V is a finite-dimensional vector space of dimension n. Every bilinear form can be represented by a matrix with respect to some ordered basis $\mathcal{B} = \{\boldsymbol{e_1}\,, \boldsymbol{e_2}\,, \ldots\,, \boldsymbol{e_n}\}$. That is, every bilinear form can be written as:

$$f(\boldsymbol{v}, \boldsymbol{w}) = A_{ij} v^i w^j\,, \tag{13}$$

where the Einstein summation convention for repeated indices is employed. In eq. (13), we have written $\boldsymbol{v} \equiv v^i \boldsymbol{e_i}$ and $\boldsymbol{w} \equiv w^j \boldsymbol{e_j}$, where v^i [w^j] are the components of the vector $\boldsymbol{v}$ [$\boldsymbol{w}$] with respect to the basis $\mathcal{B}$, and

$$A_{ij} \equiv f(\boldsymbol{e_i}\,, \boldsymbol{e_j})\,. \tag{14}$$

A_{ij} is the matrix representation of the bilinear form f with respect to the basis $\mathcal{B}$.

Suppose we choose a different basis, $\mathcal{B}' = \{\boldsymbol{f_1},\,, \boldsymbol{f_2}\,, \ldots\,, \boldsymbol{f_n}\}$, such that

$$\boldsymbol{f_j} = P^i{}_j \boldsymbol{e_i}\,,$$

for some non-singular (invertible) matrix P. If v'^j are the components of $\boldsymbol{v}$ with respect to basis $\mathcal{B}'$, then

$$\boldsymbol{v} = v'^j \boldsymbol{f_j} = v'^j P^i{}_j \boldsymbol{e_i} = v^i \boldsymbol{e_i}\,,$$

which yields

$$v^i = P^i{}_j v'^j\,.$$

Hence, it follows that

$$f(\boldsymbol{v}, \boldsymbol{w}) = A_{ij} v^i w^j = A_{ij} P^i{}_k v'^k P^j{}_\ell w'^\ell \equiv A'_{k\ell} v'^k w'^\ell\,,$$

where

$$A'_{k\ell} \equiv P^i{}_k A_{ij} P^j{}_\ell\,. \tag{15}$$

In matrix language, eq. (15) can be written as

$$A' \equiv P^{\mathsf{T}} A P\,. \tag{16}$$

Any two matrices A and A' related as in eq. (16) are said to be *congruent*.

We now specialize to skew-symmetric bilinear forms (which are called antisymmetric by physicists). A bilinear form is called *skew-symmetric* if

$$f(\boldsymbol{v}, \boldsymbol{w}) = -f(\boldsymbol{w}, \boldsymbol{v}) \quad \text{for all } \boldsymbol{v}, \boldsymbol{w} \in V\,.$$

Note that this definition implies that:

$$f(\boldsymbol{v}, \boldsymbol{v}) = 0 \quad \text{for all } \boldsymbol{v} \in V\,,$$

and vice versa.[3] If V is a finite-dimensional vector space, then a bilinear form f is skew-symmetric if and only if it is represented by an antisymmetric matrix $A^{\mathsf{T}} = -A$ with respect to any ordered basis $\mathcal{B}$. This result follows immediately from eqs. (13) and (14). In particular, eq. (16) implies that if A is antisymmetric, then so is any matrix congruent to A.

APPENDIX B: Proof of Theorem 1

In this appendix, we provide two proofs of the following theorem.

Theorem 1: If A is an even-dimensional complex invertible $2n \times 2n$ antisymmetric matrix, then there exists an invertible matrix P such that:

$$A = P^{\mathsf{T}} J P \,,$$

where the matrix J written in 2×2 block form is given by:

$$J \equiv \operatorname{diag}\underbrace{\left\{ \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \cdots, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right\}}_{n} .$$

If A is a complex singular antisymmetric $d \times d$ matrix of rank $2n$ (where d is either even or odd and $d > 2n$), then there exists an invertible $d \times d$ matrix P such that

$$A = P^{\mathsf{T}} \widetilde{J} P \,,$$

and $\widetilde{J}$ is the $d \times d$ matrix that is given in block form by

$$\widetilde{J} \equiv \left(\begin{array}{c:c} J & \mathbb{O} \\ \hdashline \mathbb{O} & \mathbb{O} \end{array} \right) , \tag{17}$$

where the $2n \times 2n$ matrix J is defined in eq. (3) and $\mathbb{O}$ is a zero matrix of the appropriate number of rows and columns.

Proof 1:[4] If $A = 0$ (corresponding to a zero bilinear form), then the theorem is trivially satisfied. Thus, we assume that f is a non-zero skew-symmetric bilinear form acting on a d-dimensional vector space V, which is represented by the $d \times d$ antisymmetric matrix A of rank $2n$ (such that $2n \leq d$) with respect to some ordered basis $\mathcal{B}$. We shall prove that there exists another ordered basis $\mathcal{B}'$ with respect to which the skew-symmetric bilinear form f is represented by J if $2n = d$ and by $\widetilde{J}$ if $2n < d$. Theorem 1 will then follow immediately from eq. (16).

[3] Note that if $f(\boldsymbol{v}, \boldsymbol{v}) = 0$ for all $\boldsymbol{v} \in V$, then it follows that $f(\boldsymbol{v}, \boldsymbol{w}) = -f(\boldsymbol{w}, \boldsymbol{v})$ for all $\boldsymbol{v}, \boldsymbol{w} \in V$. This is easily proved by noting that $0 = f(\boldsymbol{v}+\boldsymbol{w}, \boldsymbol{v}+\boldsymbol{w}) = f(\boldsymbol{v}, \boldsymbol{w}) + f(\boldsymbol{w}, \boldsymbol{v})$.

[4] This proof makes use of the material introduced in Appendix A. See, e.g., Kenneth Hoffman and Ray Kunze, *Linear Algebra* (Prentice Hall, Inc., Englewood Cliffs, N.J., 1961), section 9.3.

Since $f \neq 0$, there are vectors $\boldsymbol{v}$, $\boldsymbol{w} \in V$ such that $f(\boldsymbol{v}, \boldsymbol{w}) \neq 0$. One can normalize the vector $\boldsymbol{v}$ such that $f(\boldsymbol{v}, \boldsymbol{w}) = -f(\boldsymbol{w}, \boldsymbol{v}) = 1$. Let $\boldsymbol{x}$ be any vector in the subspace spanned by $\boldsymbol{v}$ and $\boldsymbol{w}$. Then, $\boldsymbol{x} = a\boldsymbol{v} + b\boldsymbol{w}$ for some a, $b \in \mathbb{C}$. In particular,

$$\begin{aligned} f(\boldsymbol{x}, \boldsymbol{v}) &= f(a\boldsymbol{v} + b\boldsymbol{w}, \boldsymbol{v}) = bf(\boldsymbol{w}, \boldsymbol{v}) = -b\,, \\ f(\boldsymbol{x}, \boldsymbol{w}) &= f(a\boldsymbol{v} + b\boldsymbol{w}, \boldsymbol{w}) = af(\boldsymbol{v}, \boldsymbol{w}) = a\,, \end{aligned} \tag{18}$$

so that one can write:

$$\boldsymbol{x} = f(\boldsymbol{x}, \boldsymbol{w})\boldsymbol{v} - f(\boldsymbol{x}, \boldsymbol{v})\boldsymbol{w}\,.$$

In particular, note that $\boldsymbol{v}$ and $\boldsymbol{w}$ are necessarily linearly independent, since if $\boldsymbol{x} = \boldsymbol{0}$ then $f(\boldsymbol{x}, \boldsymbol{v}) = f(\boldsymbol{x}, \boldsymbol{w}) = 0$ by the properties of the bilinear form.

Let V_1 be the two-dimensional subspace spanned by $\boldsymbol{v}$ and $\boldsymbol{w}$. Then, the matrix representation of the bilinear form restricted to the subspace V_1 is given by $\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$. Let Y be defined as the subspace of V such that

$$Y = \{\boldsymbol{y} \in V \mid f(\boldsymbol{y}, \boldsymbol{x}) = 0 \;\; \text{for every} \;\; \boldsymbol{x} \in V_1\}\,.$$

I shall now prove that $V = V_1 \oplus Y$. This requires that two conditions are satisfied: (i) $V_1 \cap Y = \{\boldsymbol{0}\}$, where $\boldsymbol{0}$ is the zero vector; and (ii) any vector $\boldsymbol{u} \in V$ can be written uniquely as $\boldsymbol{u} = \boldsymbol{v} + \boldsymbol{y}$, where $\boldsymbol{v} \in V_1$ and $\boldsymbol{y} \in Y$. Condition (i) is satisfied by the definition of Y, namely if $\boldsymbol{v}$ and $\boldsymbol{w}$ span V_1, then $\boldsymbol{y} \in Y$ implies that $f(\boldsymbol{y}, \boldsymbol{v}) = f(\boldsymbol{y}, \boldsymbol{w}) = 0$. But the only vector $\boldsymbol{y} \in V_1$ that satisfies the latter condition is the zero vector. Next, let $\boldsymbol{u} \in V$ and define:

$$\boldsymbol{y} = \boldsymbol{u} - \boldsymbol{x}\,, \qquad \text{with} \quad \boldsymbol{x} \equiv f(\boldsymbol{u}, \boldsymbol{w})\boldsymbol{v} - f(\boldsymbol{u}, \boldsymbol{v})\boldsymbol{w}\,,$$

where $\boldsymbol{v}$ and $\boldsymbol{w}$ span V_1. A simple computation shows that $f(\boldsymbol{y}, \boldsymbol{v}) = f(\boldsymbol{y}, \boldsymbol{w}) = 0$. For example,

$$f(\boldsymbol{y}, \boldsymbol{w}) = f(\boldsymbol{u}, \boldsymbol{w}) - f(\boldsymbol{x}, \boldsymbol{w}) = f(\boldsymbol{u}, \boldsymbol{w}) - f(\boldsymbol{u}, \boldsymbol{w})f(\boldsymbol{v}, \boldsymbol{w}) = f(\boldsymbol{u}, \boldsymbol{w}) - f(\boldsymbol{u}, \boldsymbol{w}) = 0\,,$$

where we have used $f(\boldsymbol{v}, \boldsymbol{w}) = 1$. Likewise,

$$f(\boldsymbol{y}, \boldsymbol{v}) = f(\boldsymbol{u}, \boldsymbol{v}) - f(\boldsymbol{x}, \boldsymbol{v}) = f(\boldsymbol{u}, \boldsymbol{v}) + f(\boldsymbol{u}, \boldsymbol{v})f(\boldsymbol{w}, \boldsymbol{v}) = f(\boldsymbol{u}, \boldsymbol{v}) - f(\boldsymbol{u}, \boldsymbol{v}) = 0\,,$$

where we have used $f(\boldsymbol{w}, \boldsymbol{v}) = -1$. These results prove that $\boldsymbol{y} \in Y$. Thus, we have shown that $\boldsymbol{u} = \boldsymbol{x} + \boldsymbol{y}$ is the unique decomposition of $\boldsymbol{u} \in V$ into the sum of two vectors $\boldsymbol{x} \in V_1$ and $\boldsymbol{y} \in Y$. Hence, $V = V_1 \oplus Y$ as advertised. With respect to the basis $\{\boldsymbol{v}\,,\, \boldsymbol{w}\,,\, \boldsymbol{y_1}\,,\, \boldsymbol{y_2}\,,\, \ldots\,,\, \boldsymbol{y_{d-2}}\}$, where $\{\boldsymbol{y_1}\,,\, \boldsymbol{y_2}\,,\, \ldots\,,\, \boldsymbol{y_{d-2}}\}$ is a basis for the subspace Y, the matrix representation of f is

$$\left(\begin{array}{cc:c} 0 & 1 & \mathbb{O}^{\mathsf{T}} \\ -1 & 0 & \mathbb{O}^{\mathsf{T}} \\ \hdashline \mathbb{O} & \mathbb{O} & B \end{array}\right), \tag{19}$$

where $\mathbb{O}$ [$\mathbb{O}^{\mathsf{T}}$] is a $(2n-2)$-dimensional column [row] vector of zeros and B is the $(d-2) \times (d-2)$ antisymmetric matrix that represents the skew-symmetric bilinear form f restricted to the subspace Y.

If B is a $(d-2) \times (d-2)$ zero matrix, then we are done and the theorem is proven. Otherwise, B represents a non-zero skew-symmetric bilinear form f restricted to the subspace Y. We can therefore repeat the above analysis starting with the subspace Y and the corresponding matrix representation B. Suppose that $d = 2n$, in which case the antisymmetric matrices A and B are invertible. Then, by induction, we will end up after n steps with the decomposition: $V = V_1 \oplus V_2 \oplus \cdots \oplus V_n$, with a corresponding basis $\mathcal{B}' = \{\boldsymbol{v_1}\,,\, \boldsymbol{w_1}\,,\, \boldsymbol{v_2}\,,\, \boldsymbol{w_2}\,,\, \ldots\,,\, \boldsymbol{v_n}\,,\, \boldsymbol{w_n}\}$, with $\boldsymbol{v_1} \equiv \boldsymbol{v}$ and $\boldsymbol{w_1} \equiv \boldsymbol{w}$ from step 1, etc. With respect to this basis, the matrix representation of the skew-symmetric bilinear form f is given by:

$$J = \left(\begin{array}{cc:cc:c:cc} 0 & 1 & & & & & \\ -1 & 0 & & & & & \\ \hdashline & & 0 & 1 & & & \\ & & -1 & 0 & & & \\ \hdashline & & & & \ddots & & \\ \hdashline & & & & & 0 & 1 \\ & & & & & -1 & 0 \end{array}\right), \tag{20}$$

where the 2×2 matrix block $\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$ appears n times along the diagonal blocks and all the elements of the off-diagonal 2×2 blocks are zero.

If $2n < d$, then after n steps, we must find that $V = V_1 \oplus V_2 \oplus \cdots \oplus V_n \oplus Z$, where the skew-symmetric bilinear form f restricted to the subspace Z vanishes exactly.[5] In this case, the matrix representation of f is given by

$$\widetilde{J} \equiv \left(\begin{array}{c:c} J & \mathbb{O} \\ \hdashline \mathbb{O} & \mathbb{O} \end{array}\right),$$

where the $2n \times 2n$ matrix J is defined in eq. (20) and $\mathbb{O}$ is the zero matrix of the appropriate number of rows and columns. Note that if d is odd, then $\widetilde{J}$ must have at least one complete row and one complete column of zeros, as expected since the determinant of any odd-dimensional antisymmetric matrix must vanish [cf. eq. (1)].

Finally, using eq. (16), we see that if $d = 2n$ then J is congruent to A, and if $d > 2n$ then $\widetilde{J}$ is congruent to A. The proof of Theorem 1 is complete.

[5] If f restricted to the subspace Z were non-vanishing, then we could carry out another step in the induction, which would contradict the assumption that the rank of A is equal to $2n$.

Proof 2:[6] For completeness, we provide a second proof of Theorem 1 based on a direct analysis of the antisymmetric matrix A. This proof makes use of the concept of the elementary row and column operations.[7] An elementary row operation consists of one of the following three operations:

1. Interchange two rows ($R_i \leftrightarrow R_j$ for $i \neq j$);
2. Multiply a given row R_i by a non-zero constant scalar ($R_i \to cR_i$ for $c \neq 0$);
3. Replace a given row R_i as follows: $R_i \to R_i + cR_j$ for $i \neq j$ and $c \neq 0$.

Each elementary row operation can be carried out by the multiplication of an appropriate non-singular matrix (called the elementary row transformation matrix) from the left.[8] Likewise, one can define elementary column operations by replacing "row" with "column" in the above. Each elementary column operation can be carried out by the multiplication of an appropriate non-singular matrix (called the elementary column transformation matrix) from the right.[8] Finally, an *elementary cogredient operation*[9] is an elementary row operation applied to a square matrix followed by the corresponding elementary column operation.[10]

The key observation is the following. If A and B are square matrices, then A is congruent to B if and only if B is obtainable from A by a sequence of elementary cogredient operations.[11] That is, an invertible matrix R exists such that $B = R^{\mathsf{T}}AR$, where R^{T} is the non-singular matrix given by the product of the elementary row operations that are employed in the sequence of elementary cogredient operations.

With this observation, it is easy to check that starting from a complex $d \times d$ antisymmetric matrix, one can apply a simple sequence of elementary cogredient operations to convert A into the form given by eq. (19), where B is a $(d-2) \times (d-2)$ complex antisymmetric matrix. (Try it!) If $B = 0$, then we are done. Otherwise, we repeat the process starting with B. Using induction, we see that the process continues until A has been converted by a sequence of elementary cogredient operations into J or $\widetilde{J}$. In particular, if the rank of A is equal to $2n$, then A will be converted into $\widetilde{J}$ after n steps. Hence, in light of the above discussion, it follows that $A = P^{\mathsf{T}}JP$, where $[P^{\mathsf{T}}]^{-1}$ is the product of all the elementary row operation matrices employed in the sequence of elementary cogredient operations used to reduce A to its canonical form given by J if $d = 2n$ or $\widetilde{J}$ if $d > 2n$. That is, Theorem 1 is proven.[12]

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[6] More details can be found in Howard Eves, *Elementary Matrix Theory* (Dover Publications, Inc., New York, 1980).

[7] These concepts are discussed in many elementary texts on matrices and linear algebra. See, e.g., Eves, op. cit.

[8] Note that elementary row and column transformation matrices are always invertible.

[9] The term *cogredient operation* employed by Eves, op. cit., is not commonly used in the modern literature. Nevertheless, I have introduced this term here as it is a convenient way to describe the sequential application of identical row and column operations.

[10] In this context, the corresponding column operation refers to carrying out the same operation on the columns that had been performed previously on the rows.

[11] This is Theorem of 5.3.4 of Eves, op. cit.

[12] For further details, see section 5.4 of Eves, op. cit.