A BASIC NOTE ON GROUP REPRESENTATIONS AND SCHUR'S LEMMA

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ABSTRACT. Here we look at some basic results from group representation theory. We discuss Schur's Lemma in the context of $\mathbb{R}[G]$-modules and note that the result is identical in the case of odd dimensional real vector spaces.

CONTENTS

1. INTRODUCTION

Group representations allow one to study an abstract group in terms of linear transformations on vector spaces. The main point of studying group representations is to reduce group theoretic problems to those of linear algebra which are well understood. The inverse direction can also be fruitful; that is, sometimes one can understand certain problems of linear algebra better by using ideas from group representation theory.

The goals of this note are two fold. First, we collect some results from group representation theory leading to the Schur's lemma and some of its consequences. Then, we consider the case of $\mathbb{R}[G]$-modules and discuss Schur's lemma in that context. The major results and developments are stated following the authors in [2] mostly. Another standard reference for group representation theory is [1].

2. DEFINITIONS

2.1. Group representations.

Definition 1. *A representation of a group G is a vector space $\mathcal{V}$ along with a group homomorphism:*
$$\rho : G \to \mathrm{GL}(\mathcal{V})$$

Remark 1. *A compact notation for a representation of a group G is:*

$$(\mathcal{V}, \rho : G \to \mathrm{GL}(\mathcal{V}))$$

In what follows, we use the simplified notation $(\mathcal{V}, \rho)$ for a representation of a group G, having the nature of the mapping ρ in mind.

Given a representation $(\mathcal{V}, \rho)$ of a group G, we say a subspace $\mathcal{U} \subseteq \mathcal{V}$ is invariant under $\rho(G)$ if,

$$\rho(g)\mathcal{U} \subseteq \mathcal{U}, \quad \forall g \in G.$$

Definition 2. *A representation, $(\mathcal{V}, \rho)$ of a group G is said to be irreducible if the only subspaces of $\mathcal{V}$ invariant under $\rho(G)$ are $\{0\}$ and $\mathcal{V}$. In other words, $(\mathcal{V}, \rho)$ is an irreducible representation of G if for any subspace $\mathcal{U} \subseteq \mathcal{V}$,*

$$\left[\rho(g)\mathcal{U} \subseteq \mathcal{U}, \quad \forall g \in G \right] \implies \mathcal{U} = \{0\} \text{ or } \mathcal{U} = \mathcal{V}.$$

Let G be a group and $\mathcal{V}$ be a vector space, we say two representations, $(\mathcal{V}, \rho)$ and $(\mathcal{V}, \sigma)$ of G are equivalent (conjugate) if there exists $T \in \mathrm{GL}(\mathcal{V})$ such that,

$$\rho(g) = T\sigma(g)T^{-1}, \quad \forall g \in G.$$

2.1.1. *Unitary representations.* Let $\mathcal{V}$ be a finite-dimensional inner product space space over $\mathbb{F}$ with $\mathbb{F} = \mathbb{C}$ or $\mathbb{R}$, and let G be a group with a representation $(\mathcal{V}, \rho)$. We say this representation is unitary if $\rho(g)$ is unitary for all $g \in G$. In case of $\mathbb{F} = \mathbb{R}$, we say the representation is orthogonal. Unitary (orthogonal) representations are of interest as it is straightforward to decompose them into irreducible representations. More detail on this will be provided later when we discuss Maschke's Theorem.

2.2. $\mathbb{F}[G]$**-modules.** Before giving the definition and technical details for $\mathbb{F}[G]$-modules, let us take a moment to look at the big picture. Consider a group G with a representation $(\mathcal{V}, \rho)$, where $\mathcal{V}$ is a vector space over $\mathbb{F}$. For each $g \in G$, we have $\rho(g) \in \mathrm{GL}(\mathcal{V})$ and as such we can define a group action $G \times \mathcal{V} \to \mathcal{V} : (g, \boldsymbol{v}) \mapsto g \cdot \boldsymbol{v}$ by,

$$g \cdot \boldsymbol{v} = \rho(g)\boldsymbol{v}.$$

Therefore, we can view a group representation as a linear action of a group on a vector space. These observations lead us to the study $\mathbb{F}[G]$-modules.

Definition 3. *Let $\mathcal{V}$ be a vector space over a base field $\mathbb{F}$, and let G be a group. Then $\mathcal{V}$ is an $\mathbb{F}[G]$-module if there exists a multiplication map $G \times \mathcal{V} \to \mathcal{V}$ defined by,*

$$(g, \boldsymbol{v}) \mapsto g \cdot \boldsymbol{v}, \quad g \in G, \boldsymbol{v} \in \mathcal{V},$$

such that the following hold: For all $\boldsymbol{u}, \boldsymbol{v} \in \mathcal{V}$, $\alpha \in \mathbb{F}$, and $g, h \in G$,

 (a) $(gh) \cdot \boldsymbol{v} = g \cdot (h \cdot \boldsymbol{v})$
 (b) $1_G \cdot \boldsymbol{v} = \boldsymbol{v}$
 (c) $g \cdot (\alpha \boldsymbol{u} + \boldsymbol{v}) = \alpha g \cdot \boldsymbol{u} + g \cdot \boldsymbol{v}$

Let $\mathcal{V}$ be an $\mathbb{F}[G]$-module; for any $g \in G$, we define $M_g : \mathcal{V} \to \mathcal{V}$, by

$$(1) \qquad\qquad\qquad\qquad M_g(\boldsymbol{v}) = g \cdot \boldsymbol{v}.$$

Let $1_\mathcal{V}$ denote the identity mapping on $\mathcal{V}$, and 1_G denote the identity element in G. The second requirement in the definition of an $\mathbb{F}[G]$-module says,

$M_{1_G} = 1_{\mathcal{V}}$ and the third requirement says that M_g is a linear map on $\mathcal{V}$ (an endomorphism of $\mathcal{V}$). Given an $\mathbb{F}[G]$-module with a basis B, sometimes it will be useful to work with the matrix of the linear transformation M_g which we denote by $[M_g]_B$.

2.2.1. *Relation to group representations.* The following basic results make the relation between $\mathbb{F}[G]$-modules and group representations clear.

Theorem 1. *Let G be a group with a representation $(\mathcal{V}, \rho)$, where $\mathcal{V}$ is a vector space over a base field $\mathbb{F}$. Then, $\mathcal{V}$ becomes an $\mathbb{F}[G]$-module if we define the multiplication $g \cdot \boldsymbol{v}$ by*

$$g \cdot \boldsymbol{v} = \rho(g)\boldsymbol{v} \quad g \in G, \boldsymbol{v} \in \mathcal{V}.$$

Proof. The result follows directly from definitions. $\qquad\square$

Theorem 2. *Given an $\mathbb{F}[G]$-module $\mathcal{V}$, we can get a representation $(\mathcal{V}, \rho)$ of G, with ρ given by,*

$$\rho(g) = M_g,$$

where M_g is defined as in (1).

2.2.2. *$\mathbb{F}[G]$-submodules and reducibility.* Let G be a group and let $\mathcal{V}$ be an $\mathbb{F}[G]$-module with a multiplication $G \times \mathcal{V} \to \mathcal{V} : (g, \boldsymbol{v}) \mapsto g \cdot \boldsymbol{v}$. A subspace $\mathcal{U} \subseteq \mathcal{V}$ is called an $\mathbb{F}[G]$-submodule of $\mathcal{V}$, if

$$(2) \qquad\qquad g \cdot \boldsymbol{u} \in \mathcal{U}, \quad \forall \boldsymbol{u} \in \mathcal{U}.$$

Remark 2. *Note that the idea of an $\mathbb{F}[G]$-submodule is directly related to the notion of an invariant subspace of a group representation. Let $\mathcal{V}$ be a an $\mathbb{F}[G]$-module. We know by Theorem 2 that with $\rho : G \to \mathrm{GL}(\mathcal{V})$ defined by*

$$\rho : g \mapsto M_g, \quad g \in G,$$

$(\mathcal{V}, \rho)$ is a representation of the group G. Now, suppose $\mathcal{U}$ is an $\mathbb{F}[G]$-submodule of $\mathcal{V}$. Then, by (2), we have

$$\rho(g)\mathcal{U} = M_g\mathcal{U} \subseteq \mathcal{U}, \quad \forall g \in G.$$

That is, $\mathcal{U}$ is invariant under $\rho(G)$.

Definition 4. *An $\mathbb{F}[G]$-module $\mathcal{V}$ is irreducible if its only $\mathbb{F}[G]$-submodules are $\{\boldsymbol{0}\}$ and $\mathcal{V}$.*

Using Remark 2, we can see the equivalence of the notion of an irreducible $\mathbb{F}[G]$-module and an irreducible representation.

Suppose $\mathcal{V}$ is a reducible n-dimensional $\mathbb{F}[G]$-module, with an $\mathbb{F}[G]$-submodule $\mathcal{U}$ of dimension $0 < k < n$. Let B_0 be a basis of $\mathcal{U}$ and extend it to a basis B of $\mathcal{V}$. Then, for every $g \in G$, the matrix of M_g with respect to the basis B is given by,

$$[M_g] = \left[\begin{array}{c|c} X_g & Y_g \\ \hline \boldsymbol{0} & Z_g \end{array} \right].$$

Where X is $k \times k$. It is simple to see that the mapping $\rho : g \mapsto X_g$ gives a representation of G on the subspaces $\mathcal{U} = \mathrm{span}\{B_0\}$.

2.2.3. $\mathbb{F}[G]$-*homomorphisms.* Let $\mathcal{V}$ and $\mathcal{W}$ be $\mathbb{F}[G]$-modules. An $\mathbb{F}[G]$-homomorphism is an $\mathbb{F}$-linear map, $\varphi : \mathcal{V} \to \mathcal{W}$ such that,

$$\varphi(g \cdot \boldsymbol{v}) = g \cdot \varphi(\boldsymbol{v}), \quad \forall \boldsymbol{v} \in \mathcal{V}, g \in G.$$

Remark 3. *Let $\mathcal{V}$ be an $\mathbb{F}[G]$-modules with group action M_g. A mapping $T : \mathcal{V} \to \mathcal{V}$ is an $\mathbb{F}[G]$-homomorphism if and only if*

$$T\left(M_g(\boldsymbol{v})\right) = M_g\left(T(\boldsymbol{v})\right), \quad \forall g \in G, \boldsymbol{v} \in \mathcal{V},$$

or equivalently,

$$M_g T = T M_g$$

Let φ be an $\mathbb{F}[G]$-homomorphism; we use $\mathrm{Ker}\{\varphi\}$ and $\mathrm{Im}\{\varphi\}$ to denote its kernel (null-space) and image. The following basic result turns out to be very useful.

Proposition 1. *Let $\mathcal{V}$ and $\mathcal{W}$ be $\mathbb{F}[G]$-modules and $\varphi : \mathcal{V} \to \mathcal{W}$ an $\mathbb{F}[G]$-homomorphism. Then,*

    (a) $\mathrm{Ker}\{\varphi\}$ *is an $\mathbb{F}[G]$-submodule of $\mathcal{V}$.*
    (b) $\mathrm{Im}\{\varphi\}$ *is an $\mathbb{F}[G]$-submodule of $\mathcal{W}$.*

Proof. The result follows directly from definitions. To see the first statement, note that $\mathrm{Ker}\{\varphi\}$ is a subspace of $\mathcal{V}$ and for all $\boldsymbol{u} \in \mathrm{Ker}\{\varphi\}$ and $g \in G$,

$$\varphi(g \cdot \boldsymbol{u}) = g \cdot \varphi(\boldsymbol{u}) = g \cdot \boldsymbol{0} = \boldsymbol{0}.$$

That is, $g \cdot \boldsymbol{u} \in \mathrm{Ker}\{\varphi\}$ for all $g \in G$ and $\boldsymbol{u} \in \mathrm{Ker}\{\varphi\}$.

As for the second statement, first recall that $\mathrm{Im}\{\varphi\}$ is a subspace of $\mathcal{W}$. Take any $\boldsymbol{w} \in \mathrm{Im}\{\varphi\}$; we know $\boldsymbol{w} = \varphi(\boldsymbol{v})$ for some $\boldsymbol{v} \in \mathcal{V}$. Then, for any $g \in G$,

$$g \cdot \boldsymbol{w} = g \cdot \varphi(\boldsymbol{v}) = \varphi(g \cdot \boldsymbol{v}),$$

and thus, $g \cdot \boldsymbol{w} \in \mathrm{Im}\{\varphi\}$ for all $g \in G$ and $\boldsymbol{w} \in \mathrm{Im}\{\varphi\}$. $\qquad \square$

2.2.4. *Direct sums of $\mathbb{F}[G]$-modules.* Let $\mathcal{V}$ be an $\mathbb{F}[G]$-module with a multiplication M_g. Suppose,

$$\mathcal{V} = \mathcal{U} \oplus \mathcal{W},$$

where $\mathcal{U}$ and $\mathcal{W}$ are $\mathbb{F}[G]$-submodules of $\mathcal{V}$. Note that if B_1 and B_2 are bases for $\mathcal{U}$ and $\mathcal{W}$ respectively, then $B = B_1 \cup B_2$ is a basis for $\mathcal{V}$ and,

$$[M_g]_B = \left[\begin{array}{c|c} [M_g]_{B_1} & \boldsymbol{0} \\ \hline \boldsymbol{0} & [M_g]_{B_2} \end{array} \right].$$

Let $\mathcal{V}$ be an $\mathbb{F}[G]$-module, and suppose,

$$\mathcal{V} = \mathcal{U}_1 \oplus \mathcal{U}_2 \oplus \cdots \oplus \mathcal{U}_r,$$

where $\mathcal{U}_j$, $j = 1, \ldots, r$ are $\mathbb{F}[G]$-submodules of $\mathcal{V}$. For each $\boldsymbol{v} \in \mathcal{V}$, there exist unique $\boldsymbol{u}_1 \in \mathcal{U}_1, \ldots, \boldsymbol{u}_r \in \mathcal{U}_r$ such that $\boldsymbol{u} = \sum_j \boldsymbol{u}_j$. We define, $P_i : \mathcal{V} \to \mathcal{U}_i$ as follows,

$$(3) \qquad\qquad\qquad P_i(\boldsymbol{v}) = \boldsymbol{u}_i.$$

The following technical result is useful.

Proposition 2. *Let $\mathcal{V}$ be an $\mathbb{F}[G]$-module, and suppose,*

$$\mathcal{V} = \mathcal{U}_1 \oplus \mathcal{U}_2 \oplus \cdots \oplus \mathcal{U}_r,$$

where $\mathcal{U}_j$, $j = 1, \ldots, r$ are $\mathbb{F}[G]$-submodules of $\mathcal{V}$. Then for each $i \in \{1, \ldots, r\}$, P_i defined as in (3) is a projections of $\mathcal{V}$ into $\mathcal{U}_i$ and also an $\mathbb{F}[G]$-homomorphism on $\mathcal{V}$.

Proof. That P_i is $\mathbb{F}$-linear and is a projection is clear. To see that P_i is an $\mathbb{F}[G]$-homomorphism note that for all $g \in G$ and $\boldsymbol{v} \in \mathcal{V}$,

$$P_i(g \cdot \boldsymbol{v}) = P_i\left(g \cdot \sum_j \boldsymbol{u}_j\right) = P_i\left(\sum_j g \cdot \boldsymbol{u}_j\right) = g \cdot \boldsymbol{u}_i = g \cdot P_i(\boldsymbol{v}).$$

$\square$

2.2.5. *Unitary $\mathbb{F}[G]$-modules.* As we defined unitary representations, we can also discuss unitary $\mathbb{F}[G]$-modules.

Definition 5. *Let $\mathcal{V}$ be an $\mathbb{F}[G]$-module with $\mathbb{F} = \mathbb{C}$ (or $\mathbb{R}$). Suppose moreover that $\mathcal{V}$ is endowed with an inner product $\langle \cdot, \cdot \rangle : \mathcal{V} \times \mathcal{V} \to \mathbb{F}$. We say $\mathcal{V}$ is a unitary (orthogonal) $\mathbb{F}[G]$-module, if the group action M_g is unitary (orthogonal) for every $g \in G$:*

$$\langle M_g(\boldsymbol{u}), M_g(\boldsymbol{v}) \rangle = \langle \boldsymbol{u}, \boldsymbol{v} \rangle, \quad \forall \boldsymbol{u}, \boldsymbol{v} \in \mathcal{V}.$$

3. Maschke's Theorem

Let G be a finite group, and say $\mathcal{V}$ is a $\mathbb{C}[G]$-module. If $\mathcal{V}$ is reducible, we know it has a non-trivial submodule. Now, one may ask the following, is it possible to reduce $\mathcal{V}$ until we are left with finitely many irreducible submodules ? Maschke's Theorem [3] is a major result which points us in that direction. We will see shortly that as a consequence of Maschke's Theorem, we can say that for a finite group G, any finite-dimensional $\mathbb{F}[G]$-module with $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$ can be reduced to a direct sum of irreducible submodules.

Theorem 3 (Maschke). *Let G be a finite group and $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$. Suppose $\mathcal{V}$ is an $\mathbb{F}[G]$-module and that $\mathcal{U}$ is an $\mathbb{F}[G]$-submodule of $\mathcal{V}$. Then, there exists an $\mathbb{F}[G]$-submodule $\mathcal{W}$ of $\mathcal{V}$ such that $\mathcal{V} = \mathcal{U} \oplus \mathcal{W}$.*

Proof. If $\mathcal{U}$ is $\{\boldsymbol{0}\}$ or $\mathcal{V}$ then there is nothing to prove; hence, suppose $\mathcal{U}$ is a non-trivial proper $\mathbb{F}[G]$-submodule. Choose any subspace $\mathcal{W}_0$ of $\mathcal{V}$ such that,

$$\mathcal{V} = \mathcal{U} \oplus \mathcal{W}_0.$$

Now, for every $\boldsymbol{v} \in \mathcal{V}$, there exist unique $\boldsymbol{u} \in \mathcal{U}$ and $\boldsymbol{w} \in \mathcal{W}_0$ such that,

$$(4) \qquad\qquad\qquad \boldsymbol{v} = \boldsymbol{u} + \boldsymbol{w}.$$

Define, $\varphi : V \to V$ by,

$$\varphi(\boldsymbol{v}) = \boldsymbol{u},$$

with $\boldsymbol{u}$ as in (4). The mapping φ is a projection of $\mathcal{V}$ onto $\mathcal{U}$; we have

$$\text{Im}\{\varphi\} = \mathcal{U} \quad \text{and} \quad \text{Ker}\{\varphi\} = \mathcal{W}_0.$$

The major part of the proof is to get an $\mathbb{F}[G]$-homomorphism out of φ which is also a projection of $\mathcal{V}$ onto $\mathcal{U}$, and use the fact that kernel of such homomorphism is an $\mathbb{F}[G]$-submodule of $\mathcal{V}$. We proceed by defining the mapping σ as follows,

$$(5) \qquad \sigma(\boldsymbol{v}) = \frac{1}{|G|} \sum_{g \in G} g^{-1} \cdot \varphi(g \cdot \boldsymbol{v})$$

Clearly, σ is an endomorphism of $\mathcal{V}$ and $\mathrm{Im}\{\sigma\} \subseteq \mathcal{U}$. Next, we show that σ is an $\mathbb{F}[G]$-homomorphism; we need to check

$$\sigma(x \cdot \boldsymbol{v}) = x \cdot \sigma(\boldsymbol{v}), \quad \forall x \in G, \boldsymbol{v} \in \mathcal{V}.$$

Note that for $x \in G$ and $\boldsymbol{v} \in \mathcal{V}$,

$$(6) \qquad \sigma(x \cdot \boldsymbol{v}) = \frac{1}{|G|} \sum_{g \in G} g^{-1} \cdot \varphi(g \cdot (x \cdot \boldsymbol{v})) = \frac{1}{|G|} \sum_{g \in G} g^{-1} \cdot \varphi((gx) \cdot \boldsymbol{v}).$$

Now, as g runs through G so does $h = gx$ and so using (6), we have,

$$\begin{aligned}
\sigma(x \cdot \boldsymbol{v}) &= \frac{1}{|G|} \sum_{h \in G} (xh^{-1}) \cdot \varphi(h \cdot \boldsymbol{v}) \\
&= \frac{1}{|G|} \sum_{h \in G} x \cdot \left[h^{-1} \cdot \varphi(h \cdot \boldsymbol{v}) \right] \\
&= x \cdot \left[\frac{1}{|G|} \sum_{h \in G} h^{-1} \cdot \varphi(h \cdot \boldsymbol{v}) \right] \\
&= x \cdot \sigma(\boldsymbol{v}),
\end{aligned}$$

so that σ is an $\mathbb{F}[G]$-homomorphism.

Next, we show that $\sigma^2 = \sigma$. For every $\boldsymbol{u} \in \mathcal{U}$ and $g \in G$, since $\mathcal{U}$ is an $\mathbb{F}[G]$-submodule, we know that $g \cdot \boldsymbol{u} \in \mathcal{U}$; thus, $\varphi(g \cdot \boldsymbol{u}) = g \cdot \boldsymbol{u}$. Then, for $\boldsymbol{u} \in \mathcal{U}$, we have

$$\begin{aligned}
\sigma(\boldsymbol{u}) &= \frac{1}{|G|} \sum_{g \in G} g^{-1} \cdot \varphi(g \cdot \boldsymbol{u}) \\
&= \frac{1}{|G|} \sum_{g \in G} g^{-1} \cdot g \cdot \boldsymbol{u} \\
&= \boldsymbol{u}.
\end{aligned}$$

Hence, for every $\boldsymbol{v} \in \mathcal{V}$ with decomposition as in (4),

$$\sigma^2(\boldsymbol{v}) = \sigma(\sigma(\boldsymbol{v})) = \sigma(\sigma(\boldsymbol{u})) = \sigma(\boldsymbol{u}) = \boldsymbol{u} = \sigma(\boldsymbol{v}).$$

Thus, σ is a projection of $\mathcal{V}$ onto $\mathcal{U}$ and $\mathrm{Im}\{\sigma\} = \mathcal{U}$. Now, let $\mathcal{W} = \mathrm{Ker}\{\sigma\}$ and note that by Proposition 1, $\mathcal{W}$ is an $\mathbb{F}[G]$-submodule of $\mathcal{V}$. Finally, we can write

$$\mathcal{V} = \mathrm{Im}\{\sigma\} \oplus \mathrm{Ker}\{\sigma\} = \mathcal{U} \oplus \mathcal{W}.$$

$\square$

Remark 4. *Apart from the case of $\mathbb{F} = \mathbb{R}$ and $\mathbb{C}$, Maschke's Theorem holds for any field of positive characteristic p (such as the finite fields) if p (necessarily prime) does not divide the order of G.*

An important consequence of Maschke's Theorem is the following:

Proposition 3. *Let G be a finite group and $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$. Suppose $\mathcal{V}$ is a finite-dimensional $\mathbb{F}[G]$-module. Then, there exist irreducible $\mathbb{F}[G]$-submodules $\mathcal{U}_1, \ldots, \mathcal{U}_r$ of $\mathcal{V}$ such that*

$$\mathcal{V} = \mathcal{U}_1 \oplus \mathcal{U}_2 \oplus \cdots \oplus \mathcal{U}_r.$$

Proof. The result follows by an induction argument on dimension of $\mathcal{V}$. $\square$

3.1. **Case of infinite groups and unitary $\mathbb{F}[G]$-modules.** Maschke's Theorem is stated for finite groups, and is not necessarily true for infinite groups. Unitary $\mathbb{F}[G]$-modules (cf. Section 2.2.5), provide an special case which we discuss in the section. The results of this subsection are stated in terms orthogonal $\mathbb{R}[G]$-modules, however, the analogous result holds for the case of unitary $\mathbb{C}[G]$-modules.

Lemma 1. *Let $\mathcal{V}$ be a finite-dimensional orthogonal $\mathbb{R}[G]$-module. Suppose $\mathcal{U}$ is an $\mathbb{R}[G]$-submodule of $\mathcal{V}$. Then, $\mathcal{U}^\perp$ is also an $\mathbb{R}[G]$-submodule of $\mathcal{V}$.*

Proof. We know the group action M_g is orthogonal for all $g \in G$, and by assumption $M_g \mathcal{U} \subseteq \mathcal{U}$. Since $\mathcal{U}$ has finite dimension, we actually have $M_g \mathcal{U} = \mathcal{U}$, and consequently $M_g^T \mathcal{U} = \mathcal{U}$ also. Therefore, for every $\boldsymbol{v} \in \mathcal{U}^\perp$ and $\boldsymbol{u} \in \mathcal{U}$,

$$\langle M_g(\boldsymbol{v}), \boldsymbol{u} \rangle = \langle \boldsymbol{v}, M_g^T(\boldsymbol{u}) \rangle = 0,$$

and so $g \cdot \boldsymbol{v} = M_g(\boldsymbol{v}) \in \mathcal{U}^\perp$ for all $g \in G$ and $\boldsymbol{v} \in \mathcal{U}^\perp$. $\square$

Using this simple result, we get the following analog of Proposition 3.

Proposition 4. *Let G be any group and suppose $\mathcal{V}$ is a finite-dimensional orthogonal $\mathbb{R}[G]$-module. Then, there exist irreducible $\mathbb{R}[G]$-submodules $\mathcal{U}_1, \ldots, \mathcal{U}_r$ of $\mathcal{V}$ such that*

$$\mathcal{V} = \mathcal{U}_1 \oplus \mathcal{U}_2 \oplus \cdots \oplus \mathcal{U}_r.$$

4. Schur's Lemma

Definition 6. *Let $\mathcal{V}$ be a vector space over a base field $\mathbb{F}$. Denote by $1_\mathcal{V}$ the identity mapping on $\mathcal{V}$. A linear map $\varphi : \mathcal{V} \to \mathcal{V}$ is called scalar if $\varphi = \alpha 1_\mathcal{V}$ for some $\alpha \in \mathbb{F}$.*

The following result, known as the Schur's Lemma [4], is an important result in representation theory.

Theorem 4 (Schur). *Let $\mathcal{V}$ and $\mathcal{W}$ be irreducible $\mathbb{C}[G]$-modules.*

   (a) *If $\varphi : \mathcal{V} \to \mathcal{W}$ is a $\mathbb{C}[G]$-homomorphism, then either φ is a $\mathbb{C}[G]$-isomorphism, or $\varphi = 0$.*
   (b) *If $\varphi : \mathcal{V} \to \mathcal{V}$ is a $\mathbb{C}[G]$-homomorphism, then φ is scalar.*

Proof. *(a)*: Suppose $\varphi(\boldsymbol{v}) \neq \boldsymbol{0}$ for some $\boldsymbol{v} \in \mathcal{V}$. Then, $\text{Im}\{\varphi\} \neq \{\boldsymbol{0}\}$. Hence, $\text{Im}\{\varphi\}$ is a non-zero $\mathbb{C}[G]$-submodule of $\mathcal{W}$ and since $\mathcal{W}$ is irreducible it follows that $\text{Im}\{\varphi\} = \mathcal{W}$; that is, φ is onto. Moreover, $\text{Ker}\{\varphi\} \neq \mathcal{V}$, and thus, since $\mathcal{V}$ is irreducible, $\text{Ker}\{\varphi\} = \{\boldsymbol{0}\}$. Therefore, φ is one-to-one also.

   (b): Let $\varphi : \mathcal{V} \to \mathcal{V}$ be a $\mathbb{C}[G]$-homomorphism. Since the base field of $\mathcal{V}$ is $\mathbb{C}$, as a linear mapping on $\mathcal{V}$, φ has an eigenvalue $\alpha \in \mathbb{C}$. Consider the mapping

$\rho = \varphi - \alpha 1_{\mathcal{V}}$. Note that ρ is also a $\mathbb{C}[G]$-homomorphism on $\mathcal{V}$, and since α is an eigenvalue of φ, it follows that $\mathrm{Ker}\{\rho\} = \mathrm{Ker}\{\varphi - \alpha 1_V\} \neq \{\mathbf{0}\}$. Consequently, $\mathrm{Ker}\{\rho\}$ is a non-zero $\mathbb{C}[G]$-submodule of $\mathcal{V}$, and thus, since $\mathcal{V}$ is irreducible, $\mathrm{Ker}\{\rho\} = \mathcal{V}$. That is,

$$\varphi(\boldsymbol{v}) - \alpha\boldsymbol{v} = \mathbf{0}, \quad \forall \boldsymbol{v} \in \mathcal{V},$$

or equivalently, $\varphi = \alpha 1_{\mathcal{V}}$. $\square$

The second part of Schur's Lemma has the following converse.

Theorem 5. *Let G be a finite group and $\mathcal{V}$ be a non-zero $\mathbb{C}[G]$-module. Suppose that every $\mathbb{C}[G]$-homomorphism from $\mathcal{V}$ to $\mathcal{V}$ is scalar. Then, $\mathcal{V}$ is irreducible.*

Proof. Suppose every $\mathbb{C}[G]$-homomorphism from $\mathcal{V}$ to $\mathcal{V}$ is scalar, but assume to the contrary that $\mathcal{V}$ is reducible. Then, $\mathcal{V}$ has a non-zero proper $\mathbb{C}[G]$-submodule $\mathcal{U}$. By Maschke's Theorem, there exists a $\mathbb{C}[G]$-submodule $\mathcal{W}$ of $\mathcal{V}$ such that,

$$\mathcal{V} = \mathcal{U} \oplus \mathcal{W}.$$

For every $\boldsymbol{v} \in \mathcal{V}$, there exist unique $\boldsymbol{u} \in \mathcal{U}$ and $\boldsymbol{w} \in \mathcal{W}$, such that $\boldsymbol{v} = \boldsymbol{u} + \boldsymbol{w}$. Define the projection $P : \mathcal{V} \to \mathcal{V}$ by

$$P(\boldsymbol{v}) = \boldsymbol{u}, \quad \forall \boldsymbol{v} \in \mathcal{V}.$$

By Proposition 2, P is a $\mathbb{C}[G]$-homomorphism of $\mathcal{V}$. However, P is clearly not scalar, and thus we have a contradiction. $\square$

Remark 5. *The above result holds also for the case of $\mathbb{R}[G]$-homomorphisms.*

Remark 6. *Schur's result holds for the case of finite and infinite groups, but its converse does not necessarily hold for infinite groups.*

The following result states Schur's result and its converse in terms of group representations.

Corollary 1. *Let G be a finite group and $\mathcal{V}$ a vector space over $\mathbb{C}$. Then, a representation $(\mathcal{V}, \rho)$ of G is irreducible if and only if every linear transformation $A : \mathcal{V} \to \mathcal{V}$ which satisfies,*

(7) $$A\rho(g) = \rho(g)A, \quad \forall g \in G$$

is scalar.

Proof. As in Theorem 1, we can turn $\mathcal{V}$ to a $\mathbb{C}[G]$-module by defining the group action $G \times \mathcal{V} \to \mathcal{V} : (g, \boldsymbol{v}) \mapsto g \cdot \boldsymbol{v}$ by,

$$g \cdot \boldsymbol{v} = \rho(g)\boldsymbol{v}, \quad g \in G, \boldsymbol{v} \in \mathcal{V}.$$

Now, a linear transformation $A : \mathcal{V} \to \mathcal{V}$ is a $\mathbb{C}[G]$-homomorphism of $\mathcal{V}$ if and only if it satisfies (7) (cf. Remark 3). Hence, the result follows from Theorem 4 and Theorem 5. $\square$

4.1. **The case of $\mathbb{R}[G]$-modules.** In this section, we discuss Schur's lemma in the case of $\mathbb{R}[G]$-modules. The first statement of Theorem 4 holds in the case of $\mathbb{R}[G]$-modules, however the second part of the result does not, because a linear operator on a vector space over reals may not have any real eigenvalues. Hence, in the case of $\mathbb{R}[G]$-modules, Schur's result is as follows:

Theorem 6. *Let $\mathcal{V}$ and $\mathcal{W}$ be irreducible $\mathbb{R}[G]$-modules.*
  (a) *If $\varphi : \mathcal{V} \to \mathcal{W}$ is an $\mathbb{R}[G]$-homomorphism, then either φ is an $\mathbb{R}[G]$-isomorphism, or $\varphi = 0$.*
  (b) *If $\varphi : \mathcal{V} \to \mathcal{V}$ is an $\mathbb{R}[G]$-homomorphism with a real eigenvalue, then φ is scalar.*

Remark 7. *If $\mathcal{V}$ is an odd-dimensional $\mathbb{R}[G]$-module, it follows that any endomorphism of $\mathcal{V}$ will have at least one real eigenvalue. Therefore, for $\mathbb{R}[G]$-modules of odd dimension, the above result gives the same exact conclusions as that of Theorem 4 (Schur's result).*

Another interesting special case is that of $\mathbb{R}[G]$-modules which are also an inner product space. Let $\mathcal{V}$ be an $\mathbb{R}[G]$-module and suppose that $\mathcal{V}$ is endowed with an inner product (we may call $\mathcal{V}$ an $\mathbb{R}[G]$-module with an inner product). We say an $\mathbb{R}[G]$-homomorphism $\varphi : \mathcal{V} \to \mathcal{V}$ is self-adjoint if,

$$\langle \varphi(\boldsymbol{u}), \boldsymbol{v} \rangle = \langle \boldsymbol{u}, \varphi(\boldsymbol{v}) \rangle, \quad \forall \boldsymbol{u}, \boldsymbol{v} \in \mathcal{V}.$$

A useful special case of Theorem 6 is as follows:

Corollary 2. *Let $\mathcal{V}$ be an irreducible $\mathbb{R}[G]$-module with an inner product. If $\varphi : \mathcal{V} \to \mathcal{V}$ is a self-adjoint $\mathbb{R}[G]$-homomorphism, then φ is scalar.*

We may state Corollary 1 in the case of real inner product spaces as follows:

Corollary 3. *Let G be a finite group and $\mathcal{V}$ an inner product space over $\mathbb{R}$. Then, a representation $(\mathcal{V}, \rho)$ of G is irreducible if and only if every self adjoint linear transformation $A : \mathcal{V} \to \mathcal{V}$ which satisfies,*

$$(8) \qquad\qquad A\rho(g) = \rho(g)A, \quad \forall g \in G$$

is scalar.

Also, in view of Remark 7, we have the following result.

Corollary 4. *Let G be a finite group and $\mathcal{V}$ an odd-dimensional vector space over $\mathbb{R}$. Then, a representation $(\mathcal{V}, \rho)$ of G is irreducible if and only if every linear transformation $A : \mathcal{V} \to \mathcal{V}$ which satisfies,*

$$(9) \qquad\qquad A\rho(g) = \rho(g)A, \quad \forall g \in G$$

is scalar.

4.2. **Unitary $\mathbb{F}[G]$-modules and the converse of Schur's lemma.** Recall that the converse to Schur's result, Theorem 5, is stated for finite groups. In the case of orthogonal $\mathbb{R}[G]$-modules (or unitary $\mathbb{C}[G]$-modules), we can use Lemma 1 to prove the following:

Proposition 5. *Let G be a group and $\mathcal{V}$ be a finite-dimensional orthogonal $\mathbb{R}[G]$-module, and suppose that every $\mathbb{R}[G]$-homomorphism from $\mathcal{V}$ to $\mathcal{V}$ is scalar. Then, $\mathcal{V}$ is irreducible.*

The same result holds if replace $\mathbb{R}$ by $\mathbb{C}$ and orthogonal by unitary in the above Proposition.

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