Clifford Algebras in Physics

Ross Greenwood

Department of Physics
University of California, Santa Cruz

PHYS 251: Group Theory in Modern Physics

Exterior Algebras

Construct an oriented geometric extent of any dim $1 \leq k \leq n$:

Exterior product $\mathbf{e}_i \wedge \mathbf{e}_j = -\mathbf{e}_j \wedge \mathbf{e}_i$

Vector Oriented length ($\mathbf{a} = a_i \, \mathbf{e}_i \in \mathbb{R}^n$)

Bivector Oriented area ($\mathbf{a} \wedge \mathbf{b} = a_i b_j \, \mathbf{e}_i \wedge \mathbf{e}_j \in \bigwedge \mathbb{R}^n$)

k-**vector** Oriented k-volume ($\mathbf{e}_1 \wedge \mathbf{e}_2 \wedge \cdots \wedge \mathbf{e}_k \in \bigwedge^{k-1} \mathbb{R}^n$)

A vector space of dimension n equipped with the exterior product generates an **exterior algebra** of dimension 2^n.

$$\Lambda(\mathbb{R}^n) = \mathbb{R} \oplus \mathbb{R}^n \oplus \bigwedge \mathbb{R}^n \oplus \bigwedge^2 \mathbb{R}^n \oplus \cdots \oplus \bigwedge^{n-1} \mathbb{R}^n$$

Volume element is an n-vector: $\mathrm{d}V = \mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \cdots \wedge \mathrm{d}x^n$

Clifford Algebras

Let V be equipped with a symmetric inner product of the form $\mathbf{a} \cdot \mathbf{b} = \frac{1}{2}\left(Q(\mathbf{a}+\mathbf{b}) - Q(\mathbf{a}) - Q(\mathbf{b})\right)$ for some quadratic form Q.

Clifford product $\quad \mathbf{ab} = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \wedge \mathbf{b}$

Such an inner product space over a field $\mathbb{K}$ equipped with the Clifford product generates a **Clifford algebra** $C\ell_n(\mathbb{K})$, which...

...is an exterior product space: $C\ell_n(\mathbb{K}) \to \bigoplus_{i=0}^{n} \bigwedge^i V$

...is a composition algebra: $|\mathbf{ab}|^2 = \mathbf{ab}\bar{\mathbf{b}}\bar{\mathbf{a}} = |\mathbf{a}|^2|\mathbf{b}|^2$

...has a set of privileged orthonormal bases

$$\mathbf{e}_i\mathbf{e}_i = Q(e_i)1 \qquad \mathbf{e}_i\mathbf{e}_j = -\mathbf{e}_j\mathbf{e}_i \mid i \neq j$$

Hodge dual: $\star\mathbf{a} = \mathbf{a}\prod_{i=1}^{n} \mathbf{e}_i$ ($k \leftrightarrow n-k$, cross product in 3D)

Geometric Algebras: $C\ell_2(\mathbb{R})$ and $C\ell_3(\mathbb{R})$

A Clifford algebra over $\mathbb{R}^n$ is called a **geometric algebra**.

Generic element of $C\ell_2(\mathbb{R})$: $\mathbf{a} = a_0 + a_1\sigma_1 + a_2\sigma_2 + a_{12}\sigma_1\sigma_2$

1	σ_1	σ_2	$\sigma_1\sigma_2$
σ_1	1	$\sigma_1\sigma_2$	σ_2
σ_2	$-\sigma_1\sigma_2$	1	$-\sigma_1$
$\sigma_1\sigma_2$	$-\sigma_2$	σ_1	-1

$C\ell_2^+(\mathbb{R}) : \{1, \sigma_1\sigma_2\} \mapsto \mathbb{C}$ $\qquad$ $C\ell_3^+(\mathbb{R}) : \{1, \sigma_1\sigma_2, \sigma_2\sigma_3, \sigma_3\sigma_1\} \mapsto \mathbb{H}$

Pauli matrices provide reps of $C\ell_2(\mathbb{R})$, $C\ell_3(\mathbb{R})$: $\sigma^i\sigma^j + \sigma^j\sigma^i = 2\delta^{ij}$

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$C\ell_3(\mathbb{R})$ in the Schrodinger-Pauli Equation

$$i\frac{\partial \psi}{\partial t} = \frac{\hat{\pi}\cdot\hat{\pi}}{2m}\psi + e\phi\psi = \frac{1}{2m}(-i\nabla - e\mathbf{A})^2\psi + e\phi\psi$$

Dot product is not algebraic; "throws away geometric information."
Promote vectors ∇ and $\mathbf{A}$ to elements of $C\ell_3(\mathbb{R})$: $\nabla \to \partial\!\!\!/ = \sigma^i\partial_i$.

$$i\frac{\partial \psi}{\partial t} = \frac{1}{2m}(-i\partial\!\!\!/ - eA\!\!\!/)^2\psi + e\phi\psi$$

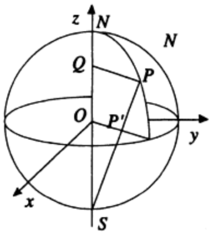
Non-commuting components $\implies$ non-vanishing exterior product

$$(i\partial\!\!\!/ + e\mathbf{A})\cdot(i\partial\!\!\!/ + e\mathbf{A}) = -\nabla^2 + ie(\partial\!\!\!/ \cdot \mathbf{A} + \mathbf{A}\cdot\nabla) + e^2A^2$$
$$(i\partial\!\!\!/ + e\mathbf{A})\wedge(i\partial\!\!\!/ + e\mathbf{A}) = -e\epsilon^{ijk}(\partial_i A_j)\sigma_k = -eB\!\!\!/$$

Scalar fields: Swap $j = 0$ irrep of $\mathfrak{su}(2)$: $\vec{\sigma} \to (0, 0, 0)$, $\sigma_i^2 = 1$.

Spinors

Stereographic projection from the unit sphere:



$$\zeta = x + iy \equiv \psi_2/\psi_1 \qquad \psi_1\psi_1^* + \psi_2\psi_2^* = 1$$

$$\frac{\psi_2'}{\psi_1'} = (1+i\theta)\frac{\psi_2}{\psi_1} = \frac{(1+i\theta/2)\psi_2}{(1-i\theta/2)\psi_1} \to \frac{e^{+i\theta/2}\,\psi_2}{e^{-i\theta/2}\,\psi_1}$$

$$x = \psi_1\psi_2^* + \psi_1^*\psi_2 \quad y = i(\psi_1\psi_2^* + \psi_1^*\psi_2) \quad z = \psi_1\psi_1^* - \psi_2\psi_2^*$$

Spinors (ψ_1, ψ_2) are elements of a complex projective space $\mathbb{CP}^n$.

Projections $\psi^\dagger \not{a} \psi = \mathrm{Tr}\, \psi\psi^\dagger \not{a} = \vec{a} \cdot \hat{\xi}$, for some $\hat{\xi}$.

Transformation $\psi^\dagger \vec{\sigma} \psi$ transforms with $\mathrm{SO}(3)$. ψ transforms with $\mathrm{Spin}(3) \cong \mathrm{SU}(2)$, the double cover of $\mathrm{SO}(3)$.

Spin Groups

A spin group is a double cover of the corresponding $\mathrm{SO}(p,q)$.

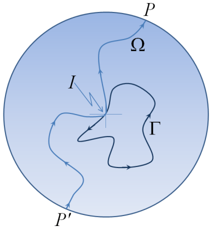
Lie Group $\mathrm{Spin}(p,q) = \{s \in C\ell_{p,q}(\mathbb{R}) \mid s^\dagger s = 1,\ \bar{s}s = 1\}$

Lie Algebra Subspace of **bivectors**: $S^{ij} = \frac{1}{4}\,[\sigma^i, \sigma^j] = \frac{1}{2}\,\sigma^i \wedge \sigma^j$

$$[S^{ij},\, S^{k\ell}] = S^{i\ell}\eta^{jk} + S^{jk}\eta^{i\ell} - S^{ik}\eta^{j\ell} - S^{j\ell}\eta^{ik}$$

Rotations $\mathbb{R}^{p+q} \to \mathbb{R}^{p+q} : \mathbf{x} \to \mathbf{x}' = s\mathbf{x}\bar{s}$

$$\mathrm{Spin}(3) \cong \mathrm{SU}(2)$$
$$\mathrm{Spin}(4) \cong \mathrm{SU}(2) \otimes \mathrm{SU}(2)$$
$$\mathrm{Spin}(1,3) \cong \mathrm{SL}(2,\mathbb{C})$$



Algebra of Physical Space

$$\sigma^\mu \equiv (1, +\vec{\sigma}) \qquad \sigma^\mu x_\mu \equiv x_0 + \vec{x}\cdot\vec{\sigma} = \begin{pmatrix} x_0 + x_3 & x_1 - ix_2 \\ x_1 + ix_2 & x_0 - x_3 \end{pmatrix}$$
$$\bar{\sigma}^\mu \equiv (1, -\vec{\sigma})$$

Lorentz Transformations $\mathrm{SO}(1,3) \mapsto \mathrm{Spin}(1,3) \cong \mathrm{SL}(2,\mathbb{C})$

$$\chi^\dagger \sigma^\mu x_\mu \chi \to \chi^\dagger \exp\left(-\tfrac{i}{2}\omega_{\mu\nu}S^{\mu\nu}\right)\sigma^\mu x_\mu \exp\left(\tfrac{i}{2}\omega_{\mu\nu}S^{\mu\nu}\right)\chi$$

Lie Algebra Subspace of **biparavectors**: $S^{\mu\nu} = \frac{1}{2}\left\langle \sigma^\mu \bar{\sigma}^\nu \right\rangle_V$

$$S^{0i} = -\tfrac{1}{2}\sigma^i = K^i \qquad S^{ij} = -\tfrac{i}{2}\epsilon_{ijk}\sigma^k = \epsilon_{ijk}J^k$$

Factorize (massless) K-G equation: $\partial^\mu\partial_\mu\chi = 0 \implies \sigma^\mu\partial_\mu\chi = 0$

Inequivalent representations σ^μ and $\bar{\sigma}^\mu$: $\left(\frac{1}{2}, 0\right)$ and $\left(0, \frac{1}{2}\right)$

Spacetime algebra $C\ell_{1,3}(\mathbb{C})$ and the Dirac Equation

$$-\hbar^2\partial^\mu\partial_\mu\psi = m^2\psi \;\rightarrow\; i\gamma^\mu\partial_\mu\psi = i\partial\!\!\!/\psi = m\psi$$

Mass occupies scalar term; time becomes a vector component γ^0.

To mix positive and negative frequencies, we need $\left(\frac{1}{2},0\right)\oplus\left(0,\frac{1}{2}\right)$.

$$S^{\mu\nu} = \frac{1}{4}\begin{pmatrix} 0 & i\sigma_1\otimes\sigma_2\sigma_1 & i\sigma_2\otimes\sigma_2\sigma_1 & i\sigma_3\otimes\sigma_2\sigma_1 \\ i\sigma_1\otimes\sigma_1\sigma_2 & 0 & -\sigma_1\sigma_2\otimes I_2 & -\sigma_1\sigma_3\otimes I_2 \\ i\sigma_2\otimes\sigma_1\sigma_2 & -\sigma_2\sigma_1\otimes I_2 & 0 & -\sigma_2\sigma_3\otimes I_2 \\ i\sigma_3\otimes\sigma_1\sigma_2 & -\sigma_3\sigma_1\otimes I_2 & -\sigma_2\sigma_3\otimes I_2 & 0 \end{pmatrix}$$

$$S^{\mu\nu} \equiv \frac{1}{2}\,\gamma^\mu\wedge\gamma^\nu \implies \gamma^0 = I_2\otimes\sigma_1, \quad \gamma^k = i\sigma_k\otimes\sigma_2$$

$$\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2\eta^{\mu\nu} I_2\otimes I_2$$

Bibliography I

- P. Lounesto. *Clifford Algebras and Spinors, Second Edition*. Cambridge University Press. (2001)
- J. Hladik. *Spinors in Physics*. Springer Science and Business Media. (2012)
- R.M. Wald. "Spinors." *General Relativity*. The University of Chicago Press. (1984)