Virasoro and Kac-Moody Algebra

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Outline

- Mathematical Description
- Conformal Symmetry in dimension $d > 3$
- Conformal Symmetry in dimension $d = 2$
- Quantum CFT

Kac-Moody algebra

Suppose g is the Lie algebra of some ordinary finite-dimensional compact connected Lie group, G. We take $\tilde{G}$ to be the set of (suitably smooth) maps from the circle S^1, to G. We could represent S^1 as the unit circle in the complex plane

$$S^1 = \{z \in \mathbb{C} : |z| = 1\} \tag{1}$$

and denote a typical map by

$$z \to \gamma(z) \in G \tag{2}$$

The group operation is defined on $\tilde{G}$ in the obvious way, given two such maps $\gamma_1, \gamma_2 : S^1 \to G$, the product of γ_1 and γ_2 is

$$\gamma_1 \cdot \gamma_2(z) = \gamma_1(z)\gamma_2(z) \tag{3}$$

This makes $\tilde{G}$ into an infinite-dimensional Lie group. It is called the loop group of G.

To find the algebra of the loop algebra, start with a basis T^a, $1 \leq a \leq$ dim g, with

$$[T^a, T^b] = if^{ab}_c T^c \tag{4}$$

A typical element of G is then of the form $\gamma = exp[-iT^a \theta_a]$
A typical element of $\tilde{G}$ can then be described by dim$g$ functions $\theta_a(z)$ defined on the unit circle,

$$\gamma(z) = exp[-iT^a \theta_a(z)] \tag{5}$$

We can make a Laurent expansion of $\theta_a(z) = \sum_{n=-\infty}^{\infty} \theta_a^{-n} z^n$, and introduce generators $T_n^a = T^a z^n$ Use the infinitesimal version of elements near identity, and commutation relation, we see that $\tilde{G}$ has the Lie algebra

$$[T_m^a, T_n^b] = if_c^{ab} T_{m+n}^c \tag{6}$$

which is called the untwisted affine Kac-Moody algebra $\tilde{g}$, the algebra of the group of maps $S^1 \rightarrow G$

Witt algebra

To construct the infinite dimensional group corresponding to the Virasoro algebra, consider the group $\tilde{V}$ of smooth one-to-one maps $S^1 \to S^1$ with the group multiplication now defined by composition

$$\xi_1 \cdot \xi_2(z) = \xi_1(\xi_2(z)) \tag{7}$$

To calculate the Lie algebra of $\tilde{V}$, consider its faithful representation defined by its action on functions $f : S^1 \to V$ where V is some vector space

$$D_\xi f(z) = f(\xi^{-1}(z)) \approx f(z) + i\epsilon(z) z \frac{d}{dz} f(z) \tag{8}$$

Making a Laurant expansion of $\epsilon(z) = \sum_{n=-\infty}^{\infty} \epsilon_{-n} z^n$, then introduce generators $L_n = -z^{n+1}\frac{d}{dz}$, We find the Lie algebra $\tilde{v}$

$$[L_m, L_n] = (m-n) L_{m+n} \tag{9}$$

which is called Witt algebra

Central Extensions

In the theory of Lie groups, Lie algebras and their representation theory, a Lie algebra extension *e* is an enlargement of a given Lie algebra *g* by another Lie algebra *h*

Let g be a Lie algebra and $\mathbb{R}^p$ an abelian Lie algebra. A Lie algebra g^0 is called a central extension of g by $\mathbb{R}^p$ if

- (i) $\mathbb{R}^p$ is (isomorphic to) an ideal contained in the center of g^0 and
- (ii) g is isomorphic to $g^0/\mathbb{R}^p$.

The central extension of Kac-Moody and Witt algebras are

$$[T_m^a, T_n^b] = if_c^{ab} T_{m+n}^c + km\delta^{ab}\delta_{m+n,0} \tag{10}$$

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}m(m^2-1)\delta_{m+n,0} \tag{11}$$

Highest weight representation

If we consider the Virasoro algebra alone, there is only one vacuum state $|h\rangle$ in any given irreducible highest weight representation and it satisfies

$$L_0 |h\rangle = h |h\rangle, \qquad L_n |h\rangle = 0, \quad n > 0 \tag{12}$$

In order for there to be a unitary representation of the Virasoro algebra corresponding to givin values of c and h, it is necessary that eithor

$$c \geq 1 \quad and \quad h \geq 0 \tag{13}$$

or

$$c = 1 - \frac{6}{(m+2)(m+3)} \tag{14}$$

and

$$h = \frac{[(m+3)p - (m+2)q]^2 - 1}{4(m+2)(m+3)} \tag{15}$$

where $\quad m = 0, 1, 2...; p = 1, 2, ..., m+1; q = 1, 2, ..., p$

Conformal Symmetry in dimension $d > 3$

By definition, a conformal transformation of the coordinates is an invertible mapping $x \to x'$ which leaves the metric tensor invariant up to a scale:

$$g'_{\mu\nu} = \Lambda(x) g_{\mu\nu}(x) \tag{16}$$

The set of conformal transformations manifestly forms a group, and it obviously has the Poincare group as a subgroup, which corresponds to the special case $\Lambda(x) = 1$

Now let's investigate the infinitesimal coordinate transformation $x^\mu \to x^\mu + \epsilon^\mu$, conformal symmetry requires that

$$\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu = f(x) g_{\mu\nu} \tag{17}$$

where

$$f(x) = \frac{2}{d} \partial_\rho \epsilon^\rho \tag{18}$$

By applying an extra derivative, we get constrains on $f(x)$

$$(2-d)\partial_\mu \partial_\nu f = \eta_{\mu\nu} \partial^2 f \tag{19}$$

The conditions require that the third derivatives of ϵ must vanish, so that it is at most quadratic in x. Here we list all the transformations

#	Finite Transformations	Generators
translation	$x'^{\mu} = x^{\mu} + a^{\mu}$	$P_{\mu} = -i\partial_{\mu}$
dilation	$x'^{\mu} = \alpha x^{\mu}$	$D = -ix_{\mu}\partial_{\mu}$
rigid rotation	$x'^{\mu} = M^{\mu}_{\nu} x^{\nu}$	$L_{\mu\nu} = i(x_{\mu}\partial_{\nu} - x_{\nu}\partial_{\mu})$
SCT	$x'^{\mu} = \frac{x^{\mu} - b^{\mu} x^2}{1 - 2b \cdot x + b^2 x^2}$	$K_{\mu} = -i(2x_{\mu}x^{\nu}\partial_{\nu} - x^2\partial_{\mu})$

Table: Conformal Symmetry Transformations

Manifestly, the special conformal transformation is a translation, preceded and followed by an inversion $x^{\mu} \to x^{\mu}/x^2$

Counting the total number of generators, we find $N = d + 1 + \frac{d(d-1)}{2} + d = \frac{(d+2)(d+1)}{2}$ And if we define the following generators

$$J_{\mu\nu} = L_{\mu\nu} \qquad J_{-1,\mu} = \frac{1}{2}(P_\mu - K_\mu) \tag{20}$$

$$J_{-1,0} = D \qquad J_{0,\mu} = \frac{1}{2}(P_\mu + K_\mu) \tag{21}$$

where $J_{ab} = -J_{ba}$ and $a, b \in \{-1, 0, 1, ..., d\}$. These new generators obey the $SO(d+1, 1)$ commutation relations

$$[J_{ab}, J_{cd}] = i(\eta_{ad}J_{bc} + \eta_{bc}J_{ad} - \eta_{ac}J_{bd} - \eta_{bd}J_{ac}) \tag{22}$$

- This shows the isomorphism between the conformal group in d dimension (Euclidean Space) and the group $SO(d+1, 1)$
- We could generalize this result, for the case of dimensions $d = p + q \geq 3$, the conformal group of $\mathbb{R}^{p,q}$ is $SO(p+1, q+1)$

Conformal Symmetry in dimension $d = 2$

The condition for invariance under infinitesimal conformal transformations in two dimensions reads as follows:

$$\partial_0 \epsilon_0 = \partial_1 \epsilon_1, \qquad \partial_0 \epsilon_1 = -\partial_1 \epsilon_0 \tag{23}$$

which we recognise as the familiar Cauchy-Riemann equations appearing in complex analysis. A complex function whose real and imaginary parts satisfy above conditions is a holomorphic function. So we can introduce complex variables in the following way

$$z = x^0 + ix^1, \qquad \epsilon = \epsilon^0 + i\epsilon^1, \qquad \partial_z = \frac{1}{2}(\partial_0 - i\partial_1), \tag{24}$$

$$\bar{z} = x^0 - ix^1, \qquad \bar{\epsilon} = \epsilon^0 - i\epsilon^1, \qquad \partial_{\bar{z}} = \frac{1}{2}(\partial_0 + i\partial_1), \tag{25}$$

Since $\epsilon(z)$ is holomorphic, so is $f(z) = z + \epsilon(z)$ which gives rise to an infinitesimal two-dimensional conformal transformation $z \to f(z)$

Since ϵ has to be holomorphic, we can perform a Laurent expansion of ϵ.

$$z' = z + \epsilon(z) = z + \sum_{n\in\mathbb{Z}} \epsilon_n z^{n+1} \tag{26}$$

$$\bar{z}' = \bar{z} + \bar{\epsilon}(\bar{z}) = \bar{z} + \sum_{n\in\mathbb{Z}} \bar{\epsilon}_n \bar{z}^{n+1} \tag{27}$$

The generators corresponding to a transformation for a particular n are

$$l_n = -z^{n+1}\partial_z \qquad \bar{l}_n = -\bar{z}^{n+1}\partial_{\bar{z}} \tag{28}$$

These generators obey the following commutation relations:

$$[l_n, l_m] = (n-m)l_{n+m} \tag{29}$$

$$[\bar{l}_n, \bar{l}_m] = (n-m)\bar{l}_{n+m} \tag{30}$$

$$[l_n, \bar{l}_m] = 0 \tag{31}$$

Thus the conformal algebra is the direct sum of two isomorphic algebras, which called Witt algebra.

Global Conformal Transformation in two dimension

Even on the Riemann sphere $S^2 \simeq \mathbb{C} \cup \{\infty\}$, $l_n, n \geq -1$ are non-singular at $z = 0$ while $l_n, n \leq 1$ are non-singular at $z = \infty$. We arrive the conclusion that
Globally defined conformal transformations on the Riemann sphere S^2 are generated by $\{l_{-1}, l_0, l_1\}$

- The operator l_{-1} generates translations $z \to z + b$
- In order to get a geometric intuition of l_0, we perform the change of variables $z = re^{i\phi}$ to find

$$l_0 + \bar{l}_0 = -r\partial_r, \qquad i(l_0 - \bar{l}_0) = -\partial_\phi, \tag{32}$$

  so $l_0 + \bar{l}_0$ is the generator for two-dimensional dilations and $i(l_0 - \bar{l}_0)$ is the generators for rotations
- The operator l_1 corresponds to special conformal transformations which are translations for the variable $w = -\frac{1}{z}$

In summary we have argued that the operators $\{l_{-1}, l_0, l_1\}$ generates transformations of the form

$$z \to \frac{az+b}{cz+d} \qquad \textit{with} \quad ad - bc = 1, \quad a, b, c, d \in \mathbb{C} \tag{33}$$

which is the Mobius group $SL(2,\mathbb{C})/Z_2 = SO(3,1)$. The Z_2 is due to the fact that the transformation is unaffected by taking all of a, b, c, d to minus themselves.
The global conformal algebra is also useful for characterizing properties of physical states. In the basis of eigenstates of the two operators l_0 and $\bar{l}_0$ are denote eigenvalues by $h, \bar{h}$. The scaling dimension Δ and the spin s of the state are given by $\Delta = h + \bar{h}$ and $s = h - \bar{h}$.

Radial quantization

In the following, we will focus our studies on conformal field theories defined on Euclidean two-dimensional flat space. We denote the Euclidean time direction by τ, and the space direction by σ.
Next, we compactify the space direction σ in a circle of unit radius. The CFT we obtain in this way is thus defined on a cylinder of infinite length. We can map this region into the whole complex plane by the mapping

$$z = exp(\tau + i\sigma) \tag{34}$$

Such a procedure of identifying dilatations with the Hamiltonian and circles about the origin with equal-time surfaces is called radial quantization.

- Since the energy-momentum tensor $T_{\alpha\beta}$ generates local translations, the dilatation current should be just $D_\alpha = T_{\alpha\beta}x^\beta$
- The statement of scale invariance implies that the energy-momentum tensor to be traceless
- With the conservation law, we arrive to the conclusion that $T_{z\bar{z}} = 0$, the field $T_{zz} \equiv T$ is an analytic function of z. Similarly $T_{\bar{z}\bar{z}} \equiv \bar{T}$ depends only on $\bar{z}$ and so is an anti-analytic field.

In radial quantization, the integral of the component of the current orthogonal to an "equal-time"(constant radius) surfaces becomes $\int j_0(x)dx \to \int j_r(\theta)d\theta$. Thus we should take

$$Q = \frac{1}{2\pi i} \oint (dz T(z)\epsilon(z) + d\bar{z}\bar{T}(\bar{z})\bar{\epsilon}(\bar{z})) \tag{35}$$

as the conserved charge. The line integral is performed over some circle of fixed radius and the integrations are taken in the counter-clockwise sense.

The variation of any filed is given by the "equal-time" commutator with the charge

$$\delta_{\epsilon,\bar{\epsilon}}\Phi(w,\bar{w}) = \frac{1}{2\pi i} \oint [dz T(z)\epsilon(z), \Phi(w,\bar{w})] + [d\bar{z}\bar{T}(\bar{z})\bar{\epsilon}(\bar{z}), \Phi(w,\bar{w})] \tag{36}$$

Primary fields

Under a general conformal transformation,

$$\Phi(z,\bar{z}) \to (\frac{\partial f}{\partial z})^h (\frac{\partial \bar{f}}{\partial \bar{z}})^{\bar{h}} \Phi(f(z), \bar{f}(\bar{z})) \tag{37}$$

where h and $\bar{h}$ are real-valued. This transformation property defines what is known as a primary field Φ of conformal weight $(h, \bar{h})$.
Not all fields in CFT will turn out to have this transformation property, the rest of the fields are known as secondary fields. Infinitesimally, under $z \to z + \epsilon(z), \bar{z} \to \bar{z} + \bar{\epsilon}(\bar{z})$, we have

$$\delta_{\epsilon,\bar{\epsilon}}\Phi(z,\bar{z}) = ((h\partial\epsilon + \epsilon\partial) + (\bar{h}\bar{\partial}\bar{\epsilon} + \bar{\epsilon}\bar{\partial}))\Phi(z,\bar{z}) \tag{38}$$

Operator Product Expansion

Compare the two infinitesimal transformation on a primary field and use residue theorem, we find the operator product expansion that defines the notion of a primary field is abbreviated as

$$T(z)\Phi(w,\bar{w}) = \frac{h}{(z-w)^2}\Phi(w,\bar{w}) + \frac{1}{z-w}\partial_w\Phi(w,\bar{w}) + \ldots \tag{39}$$

$$\bar{T}(\bar{z})\Phi(w,\bar{w}) = \frac{\bar{h}}{(\bar{z}-\bar{w})^2}\Phi(w,\bar{w}) + \frac{1}{\bar{z}-\bar{w}}\partial_{\bar{w}}\Phi(w,\bar{w}) + \ldots \tag{40}$$

and encodes the conformal transformation properties of Φ.

Central Charge and the Virasoro algebra

- Not all fields satisfy the simple transformation as a primary field. Derivatives of fields, for example, in general have more complicated transformation properties.
- A secondary field is any field that has higher than the double pole singularity in its operator product expansion with T.
- In general, the fields in a CFT can be grouped into families $[\phi_n]$ each of which contains a single primary field ϕ_n and an infinite set of secondary fields, called its descendants. These comprise the irreducible representations of the conformal group, and the primary field is the highest weight of the representation.

An example is the stress-energy tensor. By performing two conformal transformations in succession, we can determine its operator product with itself to take the form

$$T(z)T(w) = \frac{c/2}{(z-w)^4} + \frac{2}{(z-w)^2}T(w) + \frac{1}{z-w}\partial T(w) + \ldots \tag{41}$$

The constant c is known as the central charge and its value in general will depend on the particular theory under consideration.

Now define the Laurent expansion of the stress-energy tensor,

$$T(z) = \sum z^{-n-2} L_n \qquad \bar{T}(\bar{z}) = \sum \bar{z}^{-n-2} \bar{L}_n \tag{42}$$

which is formally inverted by the relations

$$L_n = \oint \frac{dz}{2\pi i} z^{n+1} T(z) \qquad \bar{L}_n = \oint \frac{d\bar{z}}{2\pi i} \bar{z}^{n+1} \bar{T}(\bar{z}) \tag{43}$$

Then we can compute the algebra of commutators satisfied by the modes L_n and $\bar{L}_n$

$$[L_n, L_m] = (\oint \frac{dz}{2\pi i} \oint \frac{dw}{2\pi i} - \frac{dw}{2\pi i} \oint \frac{dz}{2\pi i}) z^{n+1} T(z) w^{m+1} T(w) \tag{44}$$

$$= \oint \frac{dz}{2\pi i} \oint \frac{dw}{2\pi i} z^{n+1} w^{m+1} \left(\frac{c/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w} + \ldots \right) \tag{45}$$

$$= (n-m) L_{n+m} + \frac{c}{12}(n^3 - n)\delta_{n+m,0} \tag{46}$$

- The same results for $\bar{T}$, so we find two copies of Witt algebra with central extension, called the Virasoro algebra.
- Every conformally invariant quantum field theory determines a representation of this algebra with some value c and $\bar{c}$.
- We can check that the subalgebra L_{-1}, L_0, L_1 doesn't change. Thus the global conformal group remains an exact symmetry group despite the central charge.

References

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