Let A be a real or complex $n \times n$ matrix. The adjoint operator ad_A, which is a linear operator acting on the vector space of $n \times n$ matrices, is defined by

$$\mathrm{ad}_A(B) = [A, B] \equiv AB - BA\,. \tag{1}$$

Note that

$$(\mathrm{ad}_A)^n(B) = \underbrace{[A, \cdots [A, [A, B]] \cdots]}_{n} \tag{2}$$

involves n nested commutators.

Theorem 1:

$$e^A B e^{-A} = \exp(\mathrm{ad}_A)(B) \equiv \sum_{n=0}^{\infty} \frac{1}{n!}(\mathrm{ad}_A)^n(B) = B + [A, B] + \tfrac{1}{2}\,[A, [A, B]] + \cdots\,. \tag{3}$$

Proof: Define

$$B(t) \equiv e^{tA} B e^{-tA}\,, \tag{4}$$

and compute the Taylor series of $B(t)$ around the point $t = 0$. A simple computation yields $B(0) = B$ and

$$\frac{dB(t)}{dt} = Ae^{tA}Be^{-tA} - e^{tA}Be^{-tA}A = [A, B(t)] = \mathrm{ad}_A(B(t))\,. \tag{5}$$

Higher derivatives can also be computed. It is a simple exercise to show that:

$$\frac{d^n B(t)}{dt^n} = (\mathrm{ad}_A)^n(B(t))\,. \tag{6}$$

Theorem 1 then follows by substituting $t = 1$ in the resulting Taylor series expansion of $B(t)$.

We now introduce two auxiliary functions that are defined by their power series:

$$f(z) = \frac{e^z - 1}{z} = \sum_{n=0}^{\infty} \frac{z^n}{(n+1)!}\,, \qquad |z| < \infty\,, \tag{7}$$

$$g(z) = \frac{\ln z}{z - 1} = \sum_{n=0}^{\infty} \frac{(1-z)^n}{n+1}\,, \qquad |1 - z| < 1\,. \tag{8}$$

These functions satisfy:

$$f(\ln z)\, g(z) = 1\,, \qquad \text{for } |1 - z| < 1\,, \tag{9}$$

$$f(z)\, g(e^z) = 1\,, \qquad \text{for} |z| < \infty\,. \tag{10}$$

Theorem 2:

$$e^{A(t)}\frac{d}{dt}e^{-A(t)} = -f(\mathrm{ad}_A)\left(\frac{dA}{dt}\right)\,, \tag{11}$$

where $f(z)$ is defined via its Taylor series in eq. (7). Note that in general, $A(t)$ does not commute with dA/dt. A simple example, $A(t) = A + tB$ where A and B are independent of t and $[A, B] \neq 0$, illustrates this point. In the special case where $[A(t), dA/dt] = 0$, eq. (11) reduces to

$$e^{A(t)}\frac{d}{dt}e^{-A(t)} = -\frac{dA}{dt}\,, \qquad \text{if} \quad \left[A(t), \frac{dA}{dt}\right] = 0\,. \tag{12}$$

Proof: Define

$$B(s,t) \equiv e^{sA(t)}\frac{d}{dt}e^{-sA(t)}\,, \tag{13}$$

and compute the Taylor series of $B(s,t)$ around the point $s = 0$. It is straightforward to verify that $B(0,t) = 0$ and

$$\left.\frac{d^n B(s,t)}{ds^n}\right|_{s=0} = -(\mathrm{ad}_{A(t)})^{n-1}\left(\frac{dA}{dt}\right)\,, \tag{14}$$

for all positive integers n. Assembling the Taylor series for $B(s,t)$ and inserting $s = 1$ then yields Theorem 2. Note that if $[A(t), dA/dt] = 0$, then $(d^n B(s,t)/ds^n)_{s=0} = 0$ for all $n \geq 2$, and we recover the result of eq. (12).

Theorem 3:

$$\frac{d}{dt}e^{-A(t)} = -\int_0^1 e^{-sA}\,\frac{dA}{dt}\,e^{-(1-s)A}\,ds\,. \tag{15}$$

This integral representation is an alternative version of Theorem 2.

Proof: Consider

$$\begin{aligned}\frac{d}{ds}\left(e^{-sA}\,e^{-(1-s)B}\right) &= -Ae^{-sA}\,e^{-(1-s)B} + e^{-sA}\,e^{-(1-s)B}B \\ &= e^{-sA}(B-A)e^{-(1-s)B}\,. \end{aligned} \tag{16}$$

Integrate eq. (16) from $s = 0$ to $s = 1$.

$$\int_0^1 \frac{d}{ds}\left(e^{-sA}\,e^{-(1-s)B}\right) = e^{-sA}\,e^{-(1-s)B}\Big|_0^1 = e^{-A} - e^{-B}\,. \tag{17}$$

Using eq. (16), it follows that:

$$e^{-A} - e^{-B} = \int_0^1 ds\, e^{-sA}(B-A)e^{-(1-s)B}\,. \tag{18}$$

In eq. (18), we can replace $B \longrightarrow A + hB$, where h is an infinitesimal quantity:

$$e^{-A} - e^{-(A+hB)} = h\int_0^1 ds\, e^{-sA}Be^{-(1-s)(A+hB)}\,. \tag{19}$$

Taking the limit as $h \to 0$,

$$\lim_{h\to 0} \frac{1}{h}\left[e^{-(A+hB)} - e^{-A}\right] = -\int_0^1 ds\, e^{-sA} B e^{-(1-s)A} \,. \tag{20}$$

Finally, we note that the definition of the derivative can be used to write:

$$\frac{d}{dt} e^{-A(t)} = \lim_{h\to 0} \frac{e^{-A(t+h)} - e^{-A(t)}}{h} \,. \tag{21}$$

Using

$$A(t+h) = A(t) + h\frac{dA}{dt} + \mathcal{O}(h^2) \,, \tag{22}$$

it follows that:

$$\frac{d}{dt} e^{-A(t)} = \lim_{h\to 0} \frac{\exp\left[-\left(A(t) + h\dfrac{dA}{dt}\right)\right] - \exp[-A(t)]}{h} \,. \tag{23}$$

Thus, we can use the result of eq. (20) with $B = dA/dt$ to obtain

$$\frac{d}{dt} e^{-A(t)} = -\int_0^1 e^{-sA}\, \frac{dA}{dt}\, e^{-(1-s)A}\, ds \,, \tag{24}$$

which is the result quoted in Theorem 3.

Second proof of Theorem 2: One can now derive Theorem 2 directly from Theorem 3. Multiply eq. (15) by $e^{A(t)}$ to obtain:

$$e^{A(t)} \frac{d}{dt} e^{-A(t)} = -\int_0^1 e^{(1-s)A}\, \frac{dA}{dt}\, e^{-(1-s)A}\, ds \,. \tag{25}$$

Using Theorem 1 [see eq. (3)],

$$\begin{aligned} e^{A(t)} \frac{d}{dt} e^{-A(t)} &= -\int_0^1 \exp\left[\mathrm{ad}_{(1-s)A}\right] \left(\frac{dA}{dt}\right) ds \\ &= -\int_0^1 e^{(1-s)\mathrm{ad}_A} \left(\frac{dA}{dt}\right) ds \,. \end{aligned} \tag{26}$$

Changing variables $s \longrightarrow 1-s$, it follows that:

$$e^{A(t)} \frac{d}{dt} e^{-A(t)} = -\int_0^1 e^{s\,\mathrm{ad}_A} \left(\frac{dA}{dt}\right) ds \,. \tag{27}$$

The integral over s is trivial, and one finds:

$$e^{A(t)} \frac{d}{dt} e^{-A(t)} = \frac{1 - e^{\mathrm{ad}_A}}{\mathrm{ad}_A} \left(\frac{dA}{dt}\right) = -f(\mathrm{ad}_A) \left(\frac{dA}{dt}\right) \,, \tag{28}$$

which coincides with Theorem 2.

Theorem 4: The Baker-Campbell-Hausdorff (BCH) formula

$$\ln\left(e^A e^B\right) = B + \int_0^1 g\left[\exp(t\,\mathrm{ad}_A)\exp(\mathrm{ad}_B)\right](B)\,dt\,, \tag{29}$$

where $g(z)$ is defined via its Taylor series in eq. (8). Since $g(z)$ is only defined for $|1-z|<1$, it follows that the BCH formula for $\ln\left(e^A e^B\right)$ converges provided that $\|e^A e^B - I\| < 1$, where I is the identity matrix and $\|\cdots\|$ is a suitably defined matrix norm. Expanding the BCH formula, using the Taylor series definition of $g(z)$, yields:

$$e^A e^B = \exp\left(A + B + \tfrac{1}{2}[A,B] + \tfrac{1}{12}\left[A,[A,B]\right] + \tfrac{1}{12}\left[B,[B,A]\right] + \ldots\right)\,, \tag{30}$$

assuming that the resulting series is convergent. An example where the BCH series does not converge occurs for the following elements of SL(2,$\mathbb{R}$):

$$M = \begin{pmatrix} -e^{-\lambda} & 0 \\ 0 & -e^{\lambda}\end{pmatrix} = \exp\left[\lambda\begin{pmatrix}1 & 0 \\ 0 & -1\end{pmatrix}\right]\exp\left[\pi\begin{pmatrix}0 & 1 \\ -1 & 0\end{pmatrix}\right]\,, \tag{31}$$

where λ is any nonzero real number. It is easy to prove[1] that no matrix C exists such that $M = \exp C$. Nevertheless, the BCH formula is guaranteed to converge in a neighborhood of the identity of any Lie group.

Proof of the BCH formula: Define

$$C(t) = \ln(e^{tA}e^B)\,. \tag{32}$$

or equivalently,

$$e^{C(t)} = e^{tA}e^B\,. \tag{33}$$

Using Theorem 1, it follows that for any complex $n\times n$ matrix H,

$$\begin{aligned}\exp\left[\mathrm{ad}_{C(t)}\right](H) &= e^{C(t)}He^{-C(t)} = e^{tA}e^BHe^{-tA}e^{-B} \\ &= e^{tA}\left[\exp(\mathrm{ad}_B)(H)\right]e^{-tA} \\ &= \exp(\mathrm{ad}_{tA})\exp(\mathrm{ad}_B)(H)\,. \end{aligned} \tag{34}$$

[1]The characteristic equation for any 2×2 matrix A is given by

$$\lambda^2 - (\mathrm{Tr}\ A)\lambda + \det\ A = 0\,.$$

Hence, the eigenvalues of any 2×2 traceless matrix $A\in\mathfrak{sl}(2,\mathbb{R})$ [that is, A is an element of the Lie algebra of SL(2,$\mathbb{R}$)] are given by $\lambda_\pm = \pm(-\det\ \mathrm{A})^{1/2}$. Then,

$$\mathrm{Tr}\ e^A = \exp(\lambda_+) + \exp(\lambda_-) = \begin{cases} 2\cosh|\det\ A|^{1/2}\,, & \text{if } \det\ A \le 0\,, \\ 2\cos|\det\ A|^{1/2}\,, & \text{if } \det\ A > 0\,. \end{cases}$$

Thus, if $\det\ A\le 0$, then $\mathrm{Tr}\ e^A \ge 2$, and if $\det\ A > 0$, then $-2\le \mathrm{Tr}\ e^A < 2$. It follows that for any $A\in\mathfrak{sl}(2,\mathbb{R})$, $\mathrm{Tr}\ e^A \ge -2$. For the matrix M defined in eq. (31), $\mathrm{Tr}\ M = -2\cosh\lambda < -2$ for any nonzero real λ. Hence, no matrix C exists such that $M = \exp C$.

Hence, the following operator equation is valid:

$$\exp\left[\mathrm{ad}_{C(t)}\right] = \exp(t\,\mathrm{ad}_A)\exp(\mathrm{ad}_B)\,, \tag{35}$$

after noting that $\exp(\mathrm{ad}_{tA}) = \exp(t\,\mathrm{ad}_A)$. Next, we use Theorem 2 to write:

$$e^{C(t)}\frac{d}{dt}e^{-C(t)} = -f(\mathrm{ad}_{C(t)})\left(\frac{dC}{dt}\right)\,. \tag{36}$$

However, we can compute the left-hand side of eq. (36) directly:

$$e^{C(t)}\frac{d}{dt}e^{-C(t)} = e^{tA}e^{B}\frac{d}{dt}e^{-B}e^{-tA} = e^{tA}\frac{d}{dt}e^{-tA} = -A\,, \tag{37}$$

since B is independent of t, and tA commutes with $\frac{d}{dt}(tA)$. Combining the results of eqs. (36) and (37),

$$A = f(\mathrm{ad}_{C(t)})\left(\frac{dC}{dt}\right)\,. \tag{38}$$

Multiplying both sides of eq. (38) by $g(\exp\mathrm{ad}_{C(t)})$ and using eq. (10) yields:

$$\frac{dC}{dt} = g(\exp\mathrm{ad}_{C(t)})(A)\,. \tag{39}$$

Employing the operator equation, eq. (35), one may rewrite eq. (39) as:

$$\frac{dC}{dt} = g(\exp(t\,\mathrm{ad}_A)\exp(\mathrm{ad}_B))(A)\,, \tag{40}$$

which is a differential equation for $C(t)$. Integrating from $t = 0$ to $t = T$, one easily solves for C. The end result is

$$C(T) = B + \int_0^T g(\exp(t\,\mathrm{ad}_A)\exp(\mathrm{ad}_B))(A)\,dt\,, \tag{41}$$

where the constant of integration, B, has been obtained by setting $T = 0$. Finally, setting $T = 1$ in eq. (41) yields the BCH formula.

References:

The proofs of Theorems 1, 2 and 4 can be found in section 5.1 of *Symmetry Groups and Their Applications*, by Willard Miller Jr. (Academic Press, New York, 1972). The proof of Theorem 3 is based on results given in section 6.5 of *Positive Definite Matrices*, by Rajendra Bhatia (Princeton University Press, Princeton, NJ, 2007). Bhatia notes that eq. (15) has been attributed variously to Duhamel, Dyson, Feynman and Schwinger. See also R.M. Wilcox, J. Math. Phys. **8**, 962 (1967). Theorem 3 is also quoted in eq. (5.75) of *Weak Interactions and Modern Particle Theory*, by Howard Georgi (Dover Publications, Mineola, NY, 2009) [although the proof of this result is relegated to an exercise].

The proof of Theorem 2 using the results of Theorem 3 is based on my own analysis, although I would not be surprised to find this proof elsewhere in the literature. Finally, a nice discussion of the SL(2,$\mathbb{R}$) matrix that cannot be written as a single exponential can be found in section 3.4 of *Matrix Groups: An Introduction to Lie Group Theory*, by Andrew Baker (Springer-Verlag, London, UK, 2002), and in section 10.5(b) of *Group Theory in Physics*, Volume 2, by J.F. Cornwell (Academic Press, London, UK, 1984).