DUE: THURSDAY, MAY 7, 2015

ALERT: You should be ready with an initial choice for a term project topic on Thursday April 30. Feel free to consult with me on possible choices. A short written proposal (one paragraph would suffice plus references) is due on Thursday May 7.

Makeup lectures will be given on Monday April 27, Monday May 4, Monday May 11 and Monday June 1 from 4:15–6 pm in ISB 231.

1. A finite group G can be decomposed into conjugacy classes $\mathcal{C}_k$.

(a) Construct the set $\mathcal{C}'_k \equiv g\mathcal{C}_k g^{-1}$, which is obtained by replacing each element $x \in \mathcal{C}_k$ by gxg^{-1}. Prove that $\mathcal{C}'_k = \mathcal{C}_k$.

(b) Suppose that $D^{(i)}(g)$ is the ith irreducible (finite-dimensional) matrix representation of the finite group G. For a fixed class $\mathcal{C}_k$, prove that

$$\sum_{g\in\mathcal{C}_k} D^{(i)}_{j\ell}(g) = \frac{N_k}{n_i}\chi^{(i)}(\mathcal{C}_k)\delta_{j\ell}\,, \tag{1}$$

where n_i is the dimension of the ith irreducible representation of G, N_k is the number of elements in the kth conjugacy class and $\chi^{(i)}(\mathcal{C}_k)$ is the irreducible character corresponding to the kth conjugacy class.

HINT: Denoting the sum on the left hand side of eq. (1) by A_k and using the result of part (a), prove that $D^{(i)}(g)A_k = A_k D^{(i)}(g)$ for all $g \in G$. Then use Schur's second lemma.

(c) Starting from the completeness result that is satisfied by the matrix elements of the irreducible matrix representations of G and using the result of part (b), derive the completeness relation for the irreducible characters,

$$\frac{N_k}{O(G)}\sum_i \chi^{(i)}(\mathcal{C}_k)[\chi^{(i)}(\mathcal{C}_\ell)]^* = \delta_{k\ell}\,,$$

where $O(G)$ is the order of the group G (i.e. the number of elements of G), and the sum is taken over all inequivalent finite-dimensional irreducible representations.

(d) Using the orthogonality and the completeness relations satisfied by the irreducible characters, prove that the number of inequivalent irreducible representations of G is equal to the number of conjugacy classes.

2. Consider the transformations of the triangle that make up the dihedral group D_3. The elements of this group are $D_3 = \{e, r, r^2, d, rd, r^2d\}$, with the group multiplication law determined by the relations $r^3 = e$, $d^2 = e$ and $dr = r^2d$, where e is the identity element. In class, the following two-dimensional representation matrices for $r, d \in D_n$ were given,

$$r = \begin{pmatrix} \cos(2\pi/n) & -\sin(2\pi/n) \\ \sin(2\pi/n) & \cos(2\pi/n) \end{pmatrix}, \qquad d = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{2}$$

Setting $n = 3$, one can construct a two-dimensional matrix representation of D_3.

(a) Consider the six-dimensional function space W consisting of polynomials of degree 2 in two real variables (x, y):

$$f(x, y) = ax^2 + bxy + cy^2 + dx + ey + h\,, \tag{3}$$

where $a, b, \ldots, h$ are complex constants. We can view $(a, b, \ldots, h)$ as a six-dimensional vector that lives in a vector space which is isomorphic to W. If we perform a transformation of (x, y) under D_3 according to the two-dimensional representation obtained from eq (2) with $n = 3$, then the polynomial $f(x, y)$ given by eq. (3) transforms into another polynomial. That is, the vector $(a, b, \ldots, h)$ transforms under D_3 according to a six-dimensional representation. Compute the 6×6 matrices that represent the elements of D_3. Determine which irreducible representations of D_3 are contained in this six-dimensional representation and their corresponding multiplicities.

(b) Identify the irreducible invariant subspaces of W under D_3. Check that your result is consistent with the results of part (a).

(c) Consider the dihedral group D_4 treated in problem 4 of Problem Set 1. The elements of this group are $D_4 = \{e, r, r^2, r^3, d, rd, r^2d, r^3d\}$, with the group multiplication law determined by the relations $r^4 = e$, $d^2 = e$ and $dr = r^3d$, where e is the identity element. Using the two dimensional matrix representation given in eq. (2) with $n = 4$, verify that the group multiplication table of D_4 is preserved. Prove that this representation is irreducible.

(d) Construct the character table for the irreducible representations of D_4.

3. (a) Display all the standard Young tableaux of the permutation group S_4. From this result, enumerate the inequivalent irreducible representations of S_4 and specify their dimensions.

(b) Show that the normal subgroup $\{e, (12)(34), (13)(24), (14)(23)\}$ of S_4 is isomorphic to D_2.

(c) Given a normal subgroup N of a group G, a representation $D^{G/N}$ of the quotient group G/N can be *lifted* to give a representation D^G of the full group G by the following definition:

$$D^G(g) \equiv D^{G/N}(gN)\,.$$

That is, each element of the group $g \in G$ is assigned the matrix $D^{G/N}$ of the coset gN to which it belongs. Verify that if $D^{G/N}$ is a representation of G/N, then $D^G(g)$ is indeed a representation of the group G.

(d) Using the result of part (b), show that $D_3 \cong S_4/D_2$. Using the two-dimensional irreducible representation of D_3 given in class and the results of part (c), construct a two-dimensional representation of S_4 and determine its characters. Is the latter an *irreducible* representation of S_4?

(e) Using the known one-dimensional representations of S_4 and the results of parts (a) and (d), construct the character table for the group S_4. Determine any unknown entries in the character table by using the orthonormality and completeness relations for the irreducible characters. Using this technique, all entries of the character table can be uniquely determined up to a sign ambiguity in two of the entries.

(f) [*EXTRA CREDIT*] Resolve the ambiguity of part (e) by explicitly constructing the matrix representative of the transposition (1 2) corresponding to the three-dimensional irreducible representation of S_4. By taking the trace of this matrix, complete the character table of S_4.

4. Prove that if G is a finite group, then the direct product of an irreducible representation of G by a representation of G of dimension 1 is irreducible.

5. (a) Verify the following properties of the Pauli matrices $\vec{\boldsymbol{\sigma}} \equiv (\sigma_1\,, \sigma_2\,, \sigma_3)$:

$$\begin{aligned}
&(i)\ \sigma_i\sigma_j = I\delta_{ij} + i\epsilon_{ijk}\sigma_k\,,\\
&(ii)\ \sigma_2\,\vec{\boldsymbol{\sigma}}\,\sigma_2 = -\vec{\boldsymbol{\sigma}}^{\,*}\,,\\
&(iii)\ \exp(-i\theta\hat{\boldsymbol{n}}\boldsymbol{\cdot}\vec{\boldsymbol{\sigma}}/2) = I\cos(\theta/2) - i\hat{\boldsymbol{n}}\boldsymbol{\cdot}\vec{\boldsymbol{\sigma}}\sin(\theta/2)\,,
\end{aligned}$$

where I is the 2×2 identity matrix.

(b) In the angle-and-axis parameterizations of SO(3), a rotation by an angle θ about an axis that points along the unit vector $\hat{\boldsymbol{n}}$ is represented by an SO(3) matrix given by $R_{ij}(\hat{\boldsymbol{n}},\theta) = \exp(-i\theta\hat{\boldsymbol{n}}\boldsymbol{\cdot}\vec{\boldsymbol{J}})_{ij}$, with $(\hat{\boldsymbol{n}}\boldsymbol{\cdot}\vec{\boldsymbol{J}})_{ij} \equiv -i\epsilon_{ijk}n_k$. By convention, we assume that $0 \leq \theta \leq \pi$, and the axis $\hat{\boldsymbol{n}}$ can point in any direction. Evaluate R_{ij} explicitly by summing the Taylor series of the exponential, and show that

$$R_{ij}(\hat{\boldsymbol{n}},\theta) = n_in_j + (\delta_{ij} - n_in_j)\cos\theta - \epsilon_{ijk}n_k\sin\theta\,.$$

(c) Verify the formula:

$$e^{-i\theta\hat{\boldsymbol{n}}\boldsymbol{\cdot}\vec{\boldsymbol{\sigma}}/2}\,\sigma_j\,e^{i\theta\hat{\boldsymbol{n}}\boldsymbol{\cdot}\vec{\boldsymbol{\sigma}}/2} = R_{ij}(\hat{\boldsymbol{n}},\theta)\,\sigma_i\,.$$

(d) The set of matrices $\exp(-i\theta\hat{\boldsymbol{n}}\boldsymbol{\cdot}\vec{\boldsymbol{\sigma}}/2)$ constitutes the defining representation of SU(2). Prove that this representation is pseudo-real. [*HINT:* Property (ii) of part (a) is useful here.]

6. (a) A homomorphism from the vector space $\mathbb{R}^3$ to the set of traceless Hermitian 2×2 matrices is defined by $\vec{\boldsymbol{x}} \to \vec{\boldsymbol{x}} \cdot \vec{\boldsymbol{\sigma}}$. First, show that $\det(\vec{\boldsymbol{x}} \cdot \vec{\boldsymbol{\sigma}}) = -|\vec{\boldsymbol{x}}|^2$. Second, prove the identity:

$$x_i = \tfrac{1}{2}\mathrm{Tr}\,(\vec{\boldsymbol{x}} \cdot \vec{\boldsymbol{\sigma}}\,\sigma_i)\,.$$

This identity provides the inverse transformation from the set of traceless 2×2 Hermitian matrices to the vector space $\mathbb{R}^3$.

(b) Let $U \in \mathrm{SU}(2)$. Show that $U\,\vec{\boldsymbol{x}} \cdot \vec{\boldsymbol{\sigma}}\,U^{-1} = \vec{\boldsymbol{y}} \cdot \vec{\boldsymbol{\sigma}}$ for some vector $\vec{\boldsymbol{y}}$. Using the results of part (a), prove that an element of the rotation group exists such that $\vec{\boldsymbol{y}} = R\vec{\boldsymbol{x}}$ and determine an explicit form for $R \in \mathrm{SO}(3)$. Display a homomorphism from SU(2) onto SO(3) and prove that $\mathrm{SO}(3) \cong \mathrm{SU}(2)/\mathbb{Z}_2$.

(c) The Lie group $\mathrm{SU}(1\,,1)$ is defined as the group of 2×2 matrices V that satisfy $V\sigma_3 V^\dagger = \sigma_3$ and $\det V = 1$. (Note that V is *not* a unitary matrix.) The Lie group $\mathrm{SO}(2\,,1)$ is the group of transformations on vectors $\vec{\boldsymbol{x}} \in \mathbb{R}^3$ (with determinant equal to one) that preserves $x_1^2 + x_2^2 - x_3^2$. Display the homomorphism from $\mathrm{SU}(1\,,1)$ onto $\mathrm{SO}(2\,,1)$ and compare with part (b).