DUE: TUESDAY, May 26, 2015

1. The Möbius group is defined as the set of linear fractional transformations:

$$M = \left\{ m(z) = \frac{az+b}{cz+d}\,, \quad ad - bc = 1 \right\}\,,$$

where a, b, c, d and z are complex numbers.

(a) Show that the mapping $f : \mathrm{SL}(2,\mathbb{C}) \to M$ defined by:

$$f : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto m(z)$$

is a group homomorphism. [*HINT:* the multiplication law on M is defined by the composition of functions.]

(b) Prove that M is not simply connected and identify its universal covering group.

2. SO(3) can be represented by a ball of radius π with antipodal points identified. A point in the SO(3) group manifold is specified by a vector $\vec{\boldsymbol{\xi}}$ with $|\vec{\boldsymbol{\xi}}| \leq \pi$. Thus, the SO(3) manifold is parameterized by $\vec{\boldsymbol{\xi}} = (\xi, \theta, \phi)$, where (θ, ϕ) are the spherical angles and ξ is the magnitude of the vector $\vec{\boldsymbol{\xi}}$.

[*NOTE:* This is equivalent to the angle-and-axis parameterization where the rotation angle is ξ and the rotation axis, $\hat{\boldsymbol{\xi}}$, is specified by a polar angle θ and an azimuthal angle ϕ.]

(a) Show that the invariant integration measure of SO(3) is given by

$$d\mu(\vec{\boldsymbol{\xi}}) = \det c(\xi) \prod_i d\xi_i\,,$$

where the matrix elements of $c(\xi)$ are

$$c(\xi)_{nk} = \tfrac{1}{2}\epsilon_{\ell n j} R^{-1}_{\ell i} \frac{dR_{ij}}{d\xi_k}\,,$$

and $R_{ij} \equiv R_{ij}(\vec{\boldsymbol{\xi}})$ is the SO(3) matrix given in problem 5(b) of problem set 2.

(b) Evaluate the expression for $d\mu(\vec{\boldsymbol{\xi}})$ obtained in part (a) and show that

$$d\mu(\vec{\boldsymbol{\xi}}) = 2(1 - \cos\xi)\sin\theta d\theta d\phi d\xi\,.$$

HINT: First evaluate $d\mu(\vec{\boldsymbol{\xi}})$ in terms of Cartesian coordinates ξ_1, ξ_2 and ξ_3. Convert to spherical coordinate (ξ, θ, ϕ) at the very end of the calculation.

(c) Compute the total volume of SO(3). Compare this with the total volume of SU(2).

3. Consider a Lie group of transformations G acting on a manifold M. That is, for every $g \in G$, we have $gx = y$ for some $x, y \in M$.

(a) Let H be the set of all transformations in G that map a given point $x \in M$ into itself. Show that H is a subgroup. H has at least three names in the mathematical literature: the little group, the isotropy group, or the stability group of the point x.

(b) Consider the submanifold of M defined by $\{gx \mid g \in G\}$, for fixed $x \in M$. This is called the *orbit* through x with respect to G. Show that there is a one-to-one correspondence between the points of the orbit and the set of left cosets of H. Explain why we may conclude that $\{gx \mid g \in G\} = G/H$. Show that the coset space G/H is homogeneous.

(c) Prove that S^{n-1} =SO(n)/SO$(n-1)$ by considering the action of the rotation group on the point $(1, 0, 0, \ldots, 0) \in \mathbb{R}^n$.

(d) Prove that S^{2n-1} =U(n)/U$(n-1)$ by considering the action of the U(n) matrices on the point $(1, 0, 0, \ldots, 0) \in \mathbb{C}^n$.

(e) Complex projective space $\mathbb{CP}^n$ is defined as the space of complex lines in $\mathbb{C}^{n+1}$ through the origin. That is, $\mathbb{CP}^n$ consists of the set of vectors in $\mathbb{C}^{n+1}$ (omitting the zero vector) where we identify $(z_0, z_1, \ldots, z_n) \sim \lambda(z_0, z_1, \ldots, z_n)$, for any nonzero complex number λ. Without loss of generality, we can restrict our considerations to the vectors $\vec{\boldsymbol{v}} \in \mathbb{C}^{n+1}$ such that $\vec{\boldsymbol{v}} \cdot \vec{\boldsymbol{v}}^* = 1$. Show that U$(1) \otimes$ U(n) is the little group of the point $z = (1, 0, 0, \ldots, 0) \in \mathbb{CP}^n$, and that $\mathbb{CP}^n$ is the orbit through z with respect to U$(n+1)$. Conclude that $\mathbb{CP}^n$ = U$(n+1)$/U$(1) \otimes$ U(n).

(f) Real projective space $\mathbb{RP}^n$ can be defined analogously to $\mathbb{CP}^n$ of part (e) by replacing the field of complex numbers with the field of real numbers. What coset space can be identified with $\mathbb{RP}^n$?

(g) In parts (c)–(f), check that $\dim(G/H) = \dim G - \dim H$.

(h) [*EXTRA CREDIT*:] $\mathbb{CP}^n$ is a manifold of n complex (or $2n$ real) dimensions. $\mathbb{CP}^1$ is homeomorphic to which well-known two-dimensional real manifold?

4. Let A be an even-dimensional complex antisymmetric $2n \times 2n$ matrix, where n is a positive integer. We define the *pfaffian* of A, denoted by pf A, by:

$$\operatorname{pf} A = \frac{1}{2^n n!} \sum_{p \in S_{2n}} (-1)^p A_{i_1 i_2} A_{i_3 i_4} \cdots A_{i_{2n-1} i_{2n}} \,, \tag{1}$$

where the sum is taken over all permutations

$$p = \begin{pmatrix} 1 & 2 & \cdots & 2n \\ i_1 & i_2 & \cdots & i_{2n} \end{pmatrix}$$

and $(-1)^p$ is the sign of the permutation $p \in S_{2n}$. If A is an odd-dimensional complex antisymmetric matrix, the corresponding pfaffian is defined to be zero.

(a) By explicit calculation, show that[1]

$$\det A = (\text{pf}\, A)^2\,, \tag{2}$$

for any 2×2 and 4×4 complex antisymmetric matrix A.

(b) Prove that the determinant of any odd-dimensional complex antisymmetric matrix vanishes. As a result, the definition of the pfaffian in the odd-dimensional case is consistent with the result of eq. (2).

(c) Given an arbitrary $2n \times 2n$ complex matrix B and complex antisymmetric $2n \times 2n$ matrix A, use the definition of the pfaffian given in eq. (1) to prove the following identity:

$$\text{pf}\,(BAB^T) = \text{pf}\, A\, \det B\,.$$

(d) A complex $2n \times 2n$ matrix S is called *symplectic* if $S^T J S = J$, where S^T is the transpose of S and

$$J \equiv \begin{pmatrix} \mathbb{0} & \mathbb{1} \\ -\mathbb{1} & \mathbb{0} \end{pmatrix}\,,$$

where $\mathbb{1}$ is the $n \times n$ identity matrix and $\mathbb{0}$ is the $n \times n$ zero matrix. Prove that the set of $2n \times 2n$ complex symplectic matrices, denoted by $\text{Sp}(n, \mathbb{C})$, is a matrix Lie group[2] [*i.e.*, it is a topologically closed subgroup of $\text{GL}(2n, \mathbb{C})$].

(e) Prove that if S is a symplectic matrix, then $\det S = 1$.

HINT: It is very easy to prove that $\det S = \pm 1$ by taking the determinant of the equation $S^T J S = J$. To prove that there are no symplectic matrices with $\det S = -1$, use the results of part (c).

(f) Using the results of parts (d) and (e), prove that the matrix Lie groups $\text{Sp}(1, \mathbb{C})$ and $\text{SL}(2, \mathbb{C})$ are isomorphic.

5. The two-dimensional Poincaré group P(2) is the group consisting of two-dimensional Lorentz transformations [i.e., transformations on 2-vectors $\binom{ct}{x}$ which preserve $x^2 - c^2 t^2$] and translations in time and space. P(2) can be represented by 3×3 matrices acting linearly on the column vector, $\begin{pmatrix} ct \\ x \\ 1 \end{pmatrix}$, in analogy with the two-dimensional Euclidean group, E(2), worked out in class.

(a) Find the infinitesimal generators (i.e., differential operators) of the corresponding Lie algebra, $\mathfrak{p}(2)$. Work out the commutation relations of $\mathfrak{p}(2)$.

(b) Compute the Cartan-Killing form. Show that P(2) is noncompact and non-semisimple.

(c) Express the Lie algebra $\mathfrak{p}(2)$ as a semidirect sum of two abelian subalgebras.

[1] In fact, eq. (2) holds for all complex antisymmetric $2n \times 2n$ matrices, where n is any positive number. A general proof will be provided in a class handout.

[2] Warning: many authors denote the group of $2n \times 2n$ complex symplectic matrices by $\text{Sp}(2n, \mathbb{C})$.