

Notes on the spontaneous breaking of $SU(N)$ and $SO(N)$ via a second-rank tensor multiplet

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Abstract

The group theory of the spontaneous breaking of $SU(N)$ is explored. Two specific cases are analyzed in detail: (i) $SU(N)$ is broken to $SO(N)$ via a scalar field vacuum expectation value for a second-rank symmetric tensor multiplet, and (ii) $SU(2N)$ or $SU(2N+1)$ is broken to $Sp(2N)$ via a scalar field vacuum expectation value for a second-rank antisymmetric tensor multiplet. The case of the spontaneous breaking of $SO(2N)$ or $SO(2N+1)$ to $U(N)$ via a scalar field vacuum expectation value for a second-rank antisymmetric tensor multiplet is also treated.

1. Introduction

In these notes, I study the group theory of the spontaneous breaking of a global $SU(N)$ -symmetric field theory via a scalar field vacuum expectation value for second-rank tensor multiplet, $\langle \Sigma \rangle \equiv \Sigma_0$. The cases of a symmetric tensor and anti-symmetric tensor field are separately examined. I focus on one particular symmetry breaking pattern in each case corresponding to the maximal degeneracy of non-zero eigenvalues of $\Sigma_0^\dagger \Sigma_0$. The case of spontaneous breaking of $SO(2N)$ or $SO(2N+1)$ to $U(N)$ via a scalar field vacuum expectation value for a second-rank antisymmetric tensor multiplet is very similar to the corresponding breaking of $SU(2N)$ or $SU(2N+1)$. A previous analysis of these (and other) cases can be found in ref. [1].

2. Symmetry breaking via a second-rank symmetric tensor

Let Σ^{ab} be a symmetric second-rank tensor which transforms under $\text{SU}(N)$ as:

$$\Sigma^{ab} \longrightarrow U^a_c U^b_d \Sigma^{cd}, \quad (1)$$

where U is an $N \times N$ unitary matrix with unit determinant. Equivalently, in matrix form $\Sigma \longrightarrow U \Sigma U^T$, where U^T is the transpose of U . Suppose that Σ is a multiplet of scalar fields whose Lagrangian is invariant under global $\text{SU}(N)$ transformations. If Σ acquires a vacuum expectation value $\langle \Sigma \rangle \equiv \Sigma_0$, then the $\text{SU}(N)$ symmetry will be broken. If there exists a subgroup H of $\text{SU}(N)$, such that $U \Sigma_0 U^T = \Sigma_0$ for all $U \in H$, then the global $\text{SU}(N)$ symmetry is spontaneously broken to H . Writing $U = \exp i\theta_a T_a$ where the T_a are the unbroken generators (which span the unbroken subgroup H), it follows that for infinitesimal θ_a ,

$$(1 + i\theta_a T_a) \Sigma_0 (1 + i\theta_a T_a^T) = \Sigma_0, \quad (2)$$

which implies that

$$T_a \Sigma_0 + \Sigma_0 T_a^T = 0. \quad (3)$$

Thus, it is possible to find a basis for the traceless hermitian $\text{SU}(N)$ generators given by $\{T_a, X_b\}$ such that the T_a satisfy eq. (3). In this basis, the broken generators X_b are orthogonal to the T_a , that is $\text{Tr}(T_a X_b) = 0$.

The identity of the unbroken subgroup H depends on the choice of Σ_0 (which depends on the underlying dynamics responsible for the spontaneous symmetry breaking). Here, I shall consider the case of $H = \text{SO}(N)$, for which the most general form for Σ_0 is a complex symmetric $N \times N$ matrix that satisfies $\Sigma_0^\dagger \Sigma_0 = \Sigma_0 \Sigma_0^\dagger = |c|^2 I_N$, where $c \in \mathbb{C}$ and I_N is the $N \times N$ unit matrix. This is summarized by the following theorem.

Theorem: Suppose that Σ_0 is an $N \times N$ complex symmetric matrix that satisfies $\Sigma_0^\dagger \Sigma_0 = \Sigma_0 \Sigma_0^\dagger = |c|^2 I_N$ for some complex number c . Then, if the generators of $\text{SU}(N)$ in the defining (N -dimensional) representation are given by $\{T_a, X_b\}$, where the T_a and X_b are traceless hermitian $N \times N$ matrices that satisfy:

$$T_a \Sigma_0 + \Sigma_0 T_a^T = 0, \quad (4)$$

$$X_b \Sigma_0 - \Sigma_0 X_b^T = 0, \quad (5)$$

then the T_a span an unbroken $\text{SO}(N)$ Lie subalgebra, while the X_b are the broken generators that span an $\text{SU}(N)/\text{SO}(N)$ homogeneous space. Furthermore, $\text{Tr}(T_a X_b) = 0$.

Proof: First, I show that if $\Sigma_0^\dagger \Sigma_0 = \Sigma_0 \Sigma_0^\dagger = |c|^2 I_N$ and $T_a \Sigma_0 + \Sigma_0 T_a^T = 0$, then the T_a span an $\text{SO}(N)$ Lie algebra. Note that these two conditions imply:

$$|c|^2 T_a^T = -\Sigma_0^\dagger T_a \Sigma_0. \quad (6)$$

At this point, I note that the Takagi factorization [2, 3] for any complex symmetric matrix M corresponds to the statement that there exists a unitary matrix V such that VMV^T is diagonal with non-negative entries given by the positive square roots of the eigenvalues of $MM^\dagger$ (or $M^\dagger M$). In our case, this result implies that there exists a unitary matrix V such that¹

$$V \Sigma_0 V^T = c I_N. \quad (7)$$

The inverse of this result is $(V^T)^{-1} \Sigma_0^\dagger V^{-1} = c^* I_N$ (since $\Sigma_0^\dagger = |c|^2 \Sigma_0^{-1}$). I now define: $\tilde{T}_a \equiv VT_a V^{-1}$, where V is the unitary matrix appearing in eq. (7). Then, inserting this result into eq. (6), it follows that:

$$\begin{aligned} \tilde{T}_a^T &= \frac{-1}{|c|^2} (V^T)^{-1} \Sigma_0^\dagger V^{-1} \tilde{T}_a V \Sigma_0 V^T \\ &= -\tilde{T}_a. \end{aligned} \quad (8)$$

Likewise, one can use the same matrix V to define $\tilde{X} \equiv VXV^{-1}$. By an analogous computation, $|c|^2 X^T = \Sigma_0^\dagger X \Sigma_0$, which implies that $\tilde{X}_b^T = \tilde{X}_b$. Moreover, since the generators of $\text{SU}(N)$ are traceless and hermitian, it follows that the $\tilde{X}_b$ are also traceless and real.

Thus, I have exhibited a similarity transformation that transforms the basis of the Lie algebra spanned by the T_a to one that is spanned by the $\tilde{T}_a$. Since the $i\tilde{T}_a$ are real antisymmetric matrices, one immediately recognizes this Lie algebra as that of $\text{SO}(N)$. If an arbitrary element of the unbroken Lie algebra is exponentiated, it follows that $\exp i\theta_a T_a$ is related by a similarity transformation to $\exp i\theta_a \tilde{T}_a$. The latter consists of arbitrary $N \times N$ real orthogonal matrices, which implies that the $\exp i\theta_a T_a$ constitutes an N

¹Strictly speaking, the Takagi factorization yields a diagonal matrix with non-negative diagonal elements. If $c = |c|e^{2i\xi}$, one can obtain $\tilde{V} \Sigma_0 \tilde{V}^T = |c| I_N$ by taking $\tilde{V} = e^{i\xi} V$. However, this step is not necessary for the present argument.

dimensional representation that is equivalent to the N -dimensional defining representation of $\text{SO}(N)$.

Finally, I note that from $|c|^2 T_a^T = -\Sigma_0^\dagger T_a \Sigma_0$ and $|c|^2 X_b^T = \Sigma_0^\dagger X_b \Sigma_0$ it follows that $|c|^2 T_a^T X_b^T = \Sigma_0^\dagger T_a X_b \Sigma_0$ (since $\Sigma_0^\dagger \Sigma_0 = |c|^2 I_N$). Taking the trace yields $\text{Tr } T_a X_b = -\text{Tr } T_a X_b$, or equivalently $\text{Tr } T_a X_b = 0$. To show that the $\{T_a, X_b\}$ span the full $\text{SU}(N)$ Lie algebra, it is convenient to count the number of independent generators after applying the similarity transformation that converts the $\{T_a, X_b\}$ into $\{\tilde{T}_a, \tilde{X}_b\}$. I showed above that the $i\tilde{T}_a$ are real antisymmetric matrices whereas the $\tilde{X}_b$ are traceless real symmetric matrices. This implies that there are $\frac{1}{2}N(N-1)$ independent $\tilde{T}_a$ and $\frac{1}{2}N(N+1) - 1$ independent $\tilde{X}_b$ (the 1 is subtracted to account for the extra condition that the $\tilde{X}_b$ are traceless). The total number of $\text{SU}(N)$ generators is therefore $N^2 - 1$ as expected.

The above results are easily verified explicitly for $N = 3$. Consider for example a case in which

$$\Sigma_0 = c \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad (9)$$

which clearly satisfies $\Sigma_0^\dagger \Sigma_0 = \Sigma_0 \Sigma_0^\dagger = |c|^2 I$. Using the Gell-Mann matrices $\frac{1}{2}\lambda_a$ as the generators of $\text{SU}(3)$, it is easy to check that $T_a \Sigma_0 + \Sigma_0 T_a^T = 0$ implies that $T_a = c_1(\lambda_1 - \lambda_6) + c_2(\lambda_2 - \lambda_7) + c_3(\lambda_3 + \sqrt{3}\lambda_8)$, where $c_i \in \mathbb{R}$. That is, $\{\frac{1}{2}(\lambda_1 - \lambda_6), \frac{1}{2}(\lambda_2 - \lambda_7), \frac{1}{2}(\lambda_3 + \sqrt{3}\lambda_8)\}$ spans an $\text{SO}(3)$ Lie subalgebra of the $\text{SU}(3)$ Lie algebra. This matrix representation corresponds to the adjoint representation of an $\text{SU}(2)$ subalgebra, which when exponentiated yields a representation equivalent to the defining representation of $\text{SO}(3)$.

3. Symmetry breaking of $\text{SU}(2N)$ via a second-rank antisymmetric tensor

The case of symmetry breaking via an antisymmetric tensor exhibits many similar features. First, I shall consider the case of a global $\text{SU}(2N)$ symmetry group. Let Σ^{ab} be an antisymmetric second-rank tensor which transforms under $\text{SU}(2N)$ as $\Sigma \longrightarrow U \Sigma U^T$, where U is a $2N \times 2N$ unitary matrix with unit determinant. If Σ acquires a vacuum expectation value

$\langle \Sigma \rangle \equiv \Sigma_0$, then the $SU(2N)$ symmetry will be spontaneously broken. Writing $U = \exp i\theta_a T_a$ where the T_a are the unbroken generators, I again find

$$T_a \Sigma_0 + \Sigma_0 T_a^T = 0. \quad (10)$$

Thus, it is possible to find a basis for the traceless hermitian $SU(2N)$ generators given by $\{T_a, X_b\}$ such that the T_a satisfy eq. (10) and $\text{Tr}(T_a X_b) = 0$.

The unbroken subgroup H depends on the choice of Σ_0 . Here, I shall consider the case of $H = \text{Sp}(2N)$, for which the most general form for Σ_0 is a complex antisymmetric $2N \times 2N$ matrix that satisfies $\Sigma_0^\dagger \Sigma_0 = \Sigma_0 \Sigma_0^\dagger = |c|^2 I_{2N}$, where $c \in \mathbb{C}$ and I_{2N} is the $2N \times 2N$ unit matrix. This is summarized by the following theorem.

Theorem: Suppose that Σ_0 is a $2N \times 2N$ complex antisymmetric matrix that satisfies $\Sigma_0^\dagger \Sigma_0 = \Sigma_0 \Sigma_0^\dagger = |c|^2 I_{2N}$ for some complex number c . Then, if the generators of $SU(2N)$ in the defining ($2N$ -dimensional) representation are given by $\{T_a, X_b\}$, where the T_a and X_b are traceless hermitian $2N \times 2N$ matrices that satisfy:

$$T_a \Sigma_0 + \Sigma_0 T_a^T = 0, \quad (11)$$

$$X_b \Sigma_0 - \Sigma_0 X_b^T = 0, \quad (12)$$

then the T_a span an unbroken $\text{Sp}(2N)$ Lie subalgebra, while the X_b are the broken generators that span an $SU(2N)/\text{Sp}(2N)$ homogeneous space. Furthermore, $\text{Tr}(T_a X_b) = 0$.

Proof: First, I show that if $\Sigma_0^\dagger \Sigma_0 = \Sigma_0 \Sigma_0^\dagger = |c|^2 I_{2N}$ and $T_a \Sigma_0 + \Sigma_0 T_a^T = 0$, then the T_a span an $\text{Sp}(2N)$ Lie algebra. Note that these two conditions imply:

$$|c|^2 T_a^T = -\Sigma_0^\dagger T_a \Sigma_0. \quad (13)$$

For any even-dimensional complex antisymmetric matrix M , there exists a unitary matrix W such that $WMW^T = \text{diag}(\mathcal{J}_1, \mathcal{J}_2, \dots, \mathcal{J}_n)$ is block diagonal, where each block is a 2×2 matrix of the form $\mathcal{J}_n \equiv \begin{pmatrix} 0 & z_n \\ -z_n & 0 \end{pmatrix}$, where $z_n \in \mathbb{C}$ and the $|z_n|^2$ are the eigenvalues of $MM^\dagger$ (or $M^\dagger M$).² Applying this result to Σ_0 , I note that the eigenvalues of $\Sigma_0 \Sigma_0^\dagger$ are all degenerate and equal

²This result for complex antisymmetric matrices is the analog of the Takagi factorization for symmetric matrices [4]. Moreover, it is always possible to find a suitable choice for the unitary matrix W such that the z_i are real and non-negative. However, this step is not necessary for the present argument.

to $|c|^2$. Consider the matrix

$$J \equiv \begin{pmatrix} 0 & I_N \\ -I_N & 0 \end{pmatrix}, \quad (14)$$

where I_N is the $N \times N$ identity matrix. Since $JJ^\dagger = I_{2N}$, it follows that one can find unitary matrices W_1 and W_2 such that $W_1 \Sigma_0 W_1^T = c W_2 J W_2^T = \text{diag}(c\mathcal{J}, c\mathcal{J}, \dots, c\mathcal{J})$, where

$$\mathcal{J} \equiv \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \quad (15)$$

That is, the factorization of Σ_0 and cJ both yield the same block diagonal matrix consisting of N identical 2×2 blocks consisting of $c\mathcal{J}$. Thus, there exists a unitary matrix $V = W_2^{-1} W_1$ such that

$$V \Sigma_0 V^T = c J. \quad (16)$$

The inverse of this result is $(V^T)^{-1} \Sigma_0^\dagger V^{-1} = -c^* J$ (since $\Sigma_0^\dagger = |c|^2 \Sigma_0^{-1}$ and $J^{-1} = -J$). I now define $\tilde{T}_a \equiv V T_a V^{-1}$, where V is the unitary matrix appearing in eq. (16). Then, inserting this result into eq. (13), it follows that:

$$\begin{aligned} \tilde{T}_a^T &= \frac{-1}{|c|^2} (V^T)^{-1} \Sigma_0^\dagger V^{-1} \tilde{T}_a V \Sigma_0 V^T \\ &= J \tilde{T}_a J. \end{aligned} \quad (17)$$

Likewise, one can use the same matrix V to define $\widetilde{X}_b \equiv V X_b V^{-1}$. By an analogous computation, $|c|^2 X^T = \Sigma_0^\dagger X \Sigma_0$, which implies that $\widetilde{X}_b^T = -J \widetilde{X}_b J$.

Thus, I have exhibited a similarity transformation that transforms the basis of the Lie algebra spanned by the T_a to one that is spanned by the $\tilde{T}_a$. Since the $\tilde{T}_a$ are traceless hermitian matrices³ that satisfy $\tilde{T}_a^T = J \tilde{T}_a J$ [where J is defined by eq. (14)], one immediately recognizes this Lie algebra as that of $\text{Sp}(2N)$.⁴ If an arbitrary element of the unbroken Lie algebra

³Since $J^2 = -I_{2N}$, it follows from $\tilde{T}_a^T = J \tilde{T}_a J$ that $\text{Tr } T_a = 0$. This implies that the group theory for the breaking of $\text{U}(2N)$ to $\text{Sp}(2N)$ would work in almost precisely the same way with one difference—the unbroken generators X_b would not be traceless.

⁴This is actually the unitary symplectic Lie algebra, which is the compact real form of the complex symplectic Lie algebra. Some books use the notation $\text{Sp}(N)$ where I have used $\text{Sp}(2N)$. For more details, see ref. [5].

is exponentiated, it follows that $\exp i\theta_a T_a$ is related by a similarity transformation to $\exp i\theta_a \tilde{T}_a$. The latter consists of arbitrary $2N \times 2N$ unitary symplectic matrices, which implies that the $\exp i\theta_a T_a$ constitutes an $2N$ dimensional representation that is equivalent to the $2N$ -dimensional defining representation of $\text{Sp}(2N)$.

Finally, I note that from $|c|^2 T_a^T = -\Sigma_0^\dagger T_a \Sigma_0$ and $|c|^2 X_b^T = \Sigma_0^\dagger X_b \Sigma_0$ it follows that $|c|^2 T_a^T X_b^T = \Sigma_0^\dagger T_a X_b \Sigma_0$ (since $\Sigma_0^\dagger \Sigma_0 = |c|^2 I_{2N}$). Taking the trace yields $\text{Tr } T_a X_b = -\text{Tr } T_a X_b$, or equivalently $\text{Tr } T_a X_b = 0$. To show that the $\{T_a, X_b\}$ span the full $\text{SU}(2N)$ Lie algebra, it is convenient to count the number of independent generators after applying the similarity transformation that converts the $\{T_a, X_b\}$ into $\{\tilde{T}_a, \tilde{X}_b\}$. Since these are all $\text{SU}(2N)$ generators, they are traceless hermitian matrices. Moreover, I showed above that the $\tilde{T}_a$ satisfy $\tilde{T}_a = J \tilde{T}_a J$, whereas the $\tilde{X}_b$ are traceless and satisfy $\tilde{X}_b^T = -J \tilde{X}_b J$. More explicitly,

$$\tilde{T}_a = \begin{pmatrix} A & B \\ B^\dagger & -A^T \end{pmatrix}, \quad \tilde{X}_b = \begin{pmatrix} C & D \\ D^\dagger & C^T \end{pmatrix}, \quad (18)$$

where A, B, C and D are $N \times N$ complex matrices such that A and C are hermitian, B is symmetric, D is antisymmetric and $\text{Tr } C = 0$. Thus, the number of independent real parameters that describe $\tilde{T}_a$ corresponds to the number of parameters needed to define the hermitian matrix A and the complex symmetric matrix B , which is equal to $N^2 + N(N+1) = N(2N+1)$. Similarly, number of independent real parameters that describe $\tilde{X}_b$ corresponds to the number of parameters needed to define the traceless hermitian matrix C and the complex antisymmetric matrix D , which is equal to $N^2 - 1 + N(N+1) = N(2N-1) - 1$. The total number of $\text{SU}(2N)$ generators is therefore $(2N)^2 - 1$ as expected.

4. Symmetry breaking of $\text{SU}(2N+1)$ via a second-rank antisymmetric tensor

For the case of spontaneous breaking of an $\text{SU}(2N+1)$ global symmetry by a second-rank antisymmetric tensor field, the analysis of the previous section requires some modification. In this case, Σ is a $(2N+1) \times (2N+1)$ matrix, which acquires a vacuum expectation value Σ_0 . Once again, the

unbroken generators T_a satisfy:

$$T_a \Sigma_0 + \Sigma_0 T_a^T = 0. \quad (19)$$

In this case, one can find a basis for the traceless hermitian $SU(2N+1)$ generators given by $\{T_a, X_b, Y_c\}$ such that the T_a satisfy eq. (19) and $\text{Tr}(T_a X_b) = \text{Tr}(T_a Y_c) = \text{Tr}(X_b Y_c) = 0$.

I shall identify the unbroken subgroup H under the assumption that Σ_0 satisfies:

$$\Sigma_0^\dagger \Sigma_0 = \Sigma_0 \Sigma_0^\dagger = |c|^2 \begin{pmatrix} I_{2N} & 0 \\ 0 & 0 \end{pmatrix}, \quad (20)$$

where $c \in \mathbb{C}$. In this case, $H = \text{Sp}(2N)$. This is summarized by the following theorem.

Theorem: Suppose that Σ_0 is a $(2N+1) \times (2N+1)$ complex antisymmetric matrix that satisfies eq. (20) for some complex number c . Then, if the generators of $SU(2N+1)$ in the defining $[(2N+1)\text{-dimensional}]$ representation are given by $\{T_a, X_b, Y_c\}$, where the T_a , X_b and Y_c are traceless hermitian $(2N+1) \times (2N+1)$ matrices that satisfy:

$$T_a \Sigma_0 + \Sigma_0 T_a^T = 0, \quad (21)$$

$$X_b \Sigma_0 - \Sigma_0 X_b^T = 0, \quad (22)$$

$$\Sigma_0^\dagger Y_c \Sigma_0 = 0, \quad (23)$$

then the T_a span an unbroken $\text{Sp}(2N)$ Lie subalgebra, while the $\{X_b, Y_c\}$ are the broken generators that span an $SU(2N+1)/\text{Sp}(2N)$ homogeneous space. Furthermore, $\text{Tr}(T_a X_b) = \text{Tr}(T_a Y_c) = \text{Tr}(X_b Y_c) = 0$.

Proof: First, I show that if Σ_0 satisfies eq. (20) and $T_a \Sigma_0 + \Sigma_0 T_a^T = 0$, then the T_a span an $\text{Sp}(2N)$ Lie algebra. Here, I note that for any odd-dimensional complex antisymmetric matrix M , there exists a unitary matrix W such that $W M W^T = \text{diag}(\mathcal{J}_1, \mathcal{J}_2, \dots, \mathcal{J}_N, 0)$ where $\mathcal{J}_n \equiv \begin{pmatrix} 0 & z_n \\ -z_n & 0 \end{pmatrix}$, with $z_n \in \mathbb{C}$ and the $|z_n|^2$ are the eigenvalues of $M M^\dagger$ (or $M^\dagger M$) [4]. Introduce the $(2N+1) \times (2N+1)$ matrix:

$$K \equiv \begin{pmatrix} J & 0 \\ 0 & 0 \end{pmatrix}, \quad (24)$$

where J is the $2N \times 2N$ matrix given by eq. (14). The zeros shown above fill out the last row and the last column of the matrix K . Then applying the

above factorization to the antisymmetric matrices Σ_0 and K , it follows that there exists a unitary matrix V such that

$$V\Sigma_0V^T = cK. \quad (25)$$

Next, I multiply eq. (19) on the left by V and the right by V^T . Defining $\tilde{T}_a \equiv VT_aV^{-1}$ as before and using eq. (25), one easily derives:

$$K\tilde{T}_a^T = -\tilde{T}_aK. \quad (26)$$

Using the fact that $\tilde{T}_a$ is traceless and hermitian, eq. (26) has a unique solution:

$$\tilde{T}_a = \begin{pmatrix} t_a & 0 \\ 0 & 0 \end{pmatrix}, \quad (27)$$

where t_a is an $2N \times 2N$ hermitian matrix that satisfies $t_a^T = Jt_aJ$. Thus, the $\tilde{T}_a$ span an $\text{Sp}(2N)$ Lie subalgebra of the $\text{SU}(2N+1)$ Lie algebra.

Consider next the unbroken generators X_b and Y_c , which satisfy eqs. (22) and (23), and define $\tilde{X}_b \equiv VX_bV^{-1}$ and $\tilde{Y}_c \equiv VY_cV^{-1}$. Then eq. (25) implies that

$$K\tilde{X}_b^T = \tilde{X}_bK, \quad K^\dagger\tilde{Y}_cK = 0. \quad (28)$$

Using the fact that $\tilde{X}_b$ and $\tilde{Y}_c$ are traceless and hermitian, eq. (28) has a unique solution:

$$\tilde{X}_b = \begin{pmatrix} x_b & 0 \\ 0 & -\text{Tr } x_b \end{pmatrix}, \quad \tilde{Y}_c = \begin{pmatrix} 0 & y_c \\ y_c^\dagger & 0 \end{pmatrix}, \quad (29)$$

where x_b is an $2N \times 2N$ hermitian matrix that satisfies $x_b^T = -Jx_bJ$, and y_c is a complex $2N$ -dimensional column vector. From the explicit forms above, it is easy to check that $\text{Tr}(\tilde{T}_a\tilde{X}_b) = \text{Tr}(\tilde{T}_a\tilde{Y}_c) = \text{Tr}(\tilde{X}_b\tilde{Y}_c) = 0$, which implies that $\text{Tr}(T_aX_b) = \text{Tr}(T_aY_c) = \text{Tr}(X_bY_c) = 0$.

To show that the $\{T_a, X_b, Y_c\}$ span the full $\text{SU}(2N+1)$ Lie algebra, it is convenient to count the number of independent generators after applying the similarity transformation that converts the $\{T_a, X_b, Y_c\}$ into $\{\tilde{T}_a, \tilde{X}_b, \tilde{Y}_c\}$. Since these are all $\text{SU}(2N+1)$ generators, they are traceless hermitian matrices. Moreover, I showed above that the $\tilde{T}_a$, $\tilde{X}_b$ and $\tilde{Y}_c$ are given by eqs. (27) and (29), where $t_a^T = Jt_aJ$ and $x_b^T = -Jx_bJ$ are $2N \times 2N$ hermitian matrices and y_c is a complex $2N$ -dimensional vector. However, in contrast to $\tilde{X}_b$ in the last section, x_b is *not* traceless. Following the analysis at the end of the

previous section (but with $\text{Tr } x_b \neq 0$), it follows that the number of independent real parameters that describe $\tilde{T}_a$ and $\tilde{X}_b$ is given by $N(2N + 1)$ and $N(2N - 1)$, respectively. Adding this to the $4N$ parameters that describe $\tilde{Y}_c$ yields a total number of $\text{SU}(2N + 1)$ generators equal to $(2N + 1)^2 - 1$ as expected.

Finally, it is interesting to note that the generators of the type Y_c do not appear in the breaking of $\text{SU}(2N)$ to $\text{Sp}(2N)$ described in the previous section. This is easy to see by noting that in the previous section, $\Sigma_0^\dagger \Sigma_0 = |c|^2 I_{2N}$ implies that Σ_0^{-1} exists. Thus, in this case $\Sigma_0^\dagger Y_c \Sigma_0 = 0$ would imply that $Y_c = 0$. In the case of $\text{SU}(2N + 1)$ breaking, since Σ_0 is an odd-dimensional antisymmetric matrix, it follows that $\det \Sigma_0 = 0$. Thus, Σ_0^{-1} does not exist and a non-trivial solution for Y_c can arise, as we have explicitly shown above.

5. Symmetry breaking of $\text{SO}(2N)$ and $\text{SO}(2N + 1)$ via a second-rank antisymmetric tensor

The case of spontaneous breaking of $\text{SO}(2N)$ or $\text{SO}(2N + 1)$ to $\text{U}(N)$ via a scalar field vacuum expectation value for a second-rank antisymmetric tensor multiplet is very similar to the corresponding breaking of $\text{SU}(2N)$ or $\text{SU}(2N + 1)$ considered in the previous two sections. Thus, we provide a few details here. In the case of $\text{SO}(2N)$ the relevant theorem is as follows.

Theorem: Suppose that Σ_0 is a $2N \times 2N$ real antisymmetric matrix that satisfies $\Sigma_0^T \Sigma_0 = \Sigma_0 \Sigma_0^T = c^2 I_{2N}$ for some real number c . Then, if the generators of $\text{SO}(2N)$ in the defining ($2N$ -dimensional) representation are given by $\{T_a, X_b\}$, where the iT_a and iX_b are real antisymmetric $2N \times 2N$ matrices that satisfy:

$$T_a \Sigma_0 + \Sigma_0 T_a^T = 0, \quad (30)$$

$$X_b \Sigma_0 - \Sigma_0 X_b^T = 0, \quad (31)$$

then the T_a span an unbroken $\text{U}(N)$ Lie subalgebra, while the X_b are the broken generators that span an $\text{SO}(2N)/\text{U}(N)$ homogeneous space. Furthermore, $\text{Tr } (T_a X_b) = 0$.

Proof: First, I show that if $\Sigma_0^T \Sigma_0 = \Sigma_0 \Sigma_0^T = c^2 I_{2N}$ and $T_a \Sigma_0 + \Sigma_0 T_a^T = 0$, then the T_a span an $\text{U}(N)$ Lie subalgebra. Note that these two conditions

imply:

$$c^2 T_a^T = -\Sigma_0^T T_a \Sigma_0. \quad (32)$$

For any even-dimensional real antisymmetric matrix M , there exists a real orthogonal matrix W such that $W M W^T = \text{diag}(\mathcal{J}_1, \mathcal{J}_2, \dots, \mathcal{J}_n)$ is block diagonal, where each block is a 2×2 matrix of the form $\mathcal{J}_n \equiv \begin{pmatrix} 0 & z_n \\ -z_n & 0 \end{pmatrix}$, where $z_n \in \mathbb{R}$ and the z_n^2 are the eigenvalues of $M M^T$ (or $M^T M$).⁵ Applying this result to Σ_0 , note that the eigenvalues of $\Sigma_0 \Sigma_0^T$ are all degenerate and equal to c^2 . Moreover, since the matrix J [eq. (14)] satisfies $J J^T = I_{2N}$, it follows that one can find real orthogonal matrices W_1 and W_2 such that $W_1 \Sigma_0 W_1^T = c W_2 J W_2^T = \text{diag}(c \mathcal{J}, c \mathcal{J}, \dots, c \mathcal{J})$, where $\mathcal{J}$ is defined in eq. (15). That is, the factorization of Σ_0 and cJ both yield the same block diagonal matrix consisting of N identical 2×2 blocks consisting of $c \mathcal{J}$. Thus, there exists a real orthogonal matrix $V = W_2^{-1} W_1$ such that $V \Sigma_0 V^T = c J$. The inverse of this result is $V \Sigma_0^T V^T = -c J$ (since $J^T = -J$). I now define $\tilde{T}_a \equiv V T_a V^T$. Then eq. (32) implies that

$$\tilde{T}_a^T = \frac{-1}{c^2} V \Sigma_0^T V^T \tilde{T}_a V \Sigma_0 V^T = J \tilde{T}_a J. \quad (33)$$

Likewise, one can use the same matrix V to define $\tilde{X}_b \equiv V X_b V^T$. By an analogous computation, $c^2 X^T = \Sigma_0^T X \Sigma_0$, which implies that $\tilde{X}_b^T = -J \tilde{X}_b J$.

Recall that T_a and X_b are both antisymmetric $2N \times 2N$ matrices. Then, $\tilde{T}_a \equiv V T_a V^T$ and $\tilde{X}_a \equiv V X_a V^T$ are also antisymmetric. Hence, it follows that

$$\tilde{T}_a = -J \tilde{T}_a J, \quad \tilde{X}_a = J \tilde{X}_a J. \quad (34)$$

Using the explicit form for J , eq. (34) implies that T_a and X_b take the following block form:

$$i \tilde{T}_a = \begin{pmatrix} A & B \\ -B & A \end{pmatrix}, \quad i \tilde{X}_b = \begin{pmatrix} C & D \\ D & -C \end{pmatrix}, \quad (35)$$

where A , B , C and D are $N \times N$ real matrices such that A , C and D are antisymmetric and B is symmetric. Thus, I have exhibited a similarity transformation (note that $V^T = V^{-1}$) that transforms the basis of the Lie algebra spanned by the T_a to one that is spanned by the $\tilde{T}_a$. Moreover, consider the isomorphism that maps $i \tilde{T}_a$ given in eq. (35) to the $N \times N$

⁵This result for real antisymmetric matrices is the analog of the corresponding factorization of complex antisymmetric matrices quoted in Section 3.

matrix $A + iB$. Since $(A + iB)^\dagger = (A - iB)^T = -(A + iB)$, we see that the $A + iB$ are anti-hermitian generators (which are not generally traceless) that span a $U(N)$ subalgebra of the $SO(2N)$. We can check the number of unbroken generators by counting the number of degrees of freedom of one real antisymmetric and one real symmetric matrix: $\frac{1}{2}N(N-1) + \frac{1}{2}N(N+1) = N^2$, as expected.

Finally, I note that from $c^2 T_a^T = -\Sigma_0^T T_a \Sigma_0$ and $c^2 X_b^T = \Sigma_0^T X_b \Sigma_0$ it follows that $c^2 T_a^T X_b^T = \Sigma_0^T T_a X_b \Sigma_0$ (since $\Sigma_0^T \Sigma_0 = c^2 I_{2N}$). Taking the trace yields $\text{Tr } T_a X_b = -\text{Tr } T_a X_b$, or equivalently $\text{Tr } T_a X_b = 0$. To show that the $\{T_a, X_b\}$ span the full $SO(2N)$ Lie algebra, we note that there are N^2 unbroken generators and $N(N-1)$ broken generators (corresponding to the number of parameters describing two real antisymmetric matrices [see eq. (35)]). Thus, the total number of generators is $N(2N-1)$ which matches the total number of $SO(2N)$ generators.

Finally, we turn to the case of $SO(2N+1)$ breaking. In this case, we will make use of the fact that for any odd-dimensional real antisymmetric matrix M , there exists a real orthogonal matrix W such that $WMW^T = \text{diag}(\mathcal{J}_1, \mathcal{J}_2, \dots, \mathcal{J}_N, 0)$ where $\mathcal{J}_n \equiv \begin{pmatrix} 0 & z_n \\ -z_n & 0 \end{pmatrix}$, with $z_n \in \mathbb{R}$ and the z_n^2 are the eigenvalues of MM^T (or $M^T M$). The relevant theorem for case of $SO(2N+1)$ is as follows.

Theorem: Suppose that Σ_0 is a $(2N+1) \times (2N+1)$ real antisymmetric matrix that satisfies

$$\Sigma_0^T \Sigma_0 = \Sigma_0 \Sigma_0^T = c^2 \begin{pmatrix} I_{2N} & 0 \\ 0 & 0 \end{pmatrix}, \quad (36)$$

where $c \in \mathbb{R}$. Then, if the generators of $SO(2N+1)$ in the defining $[(2N+1)\text{-dimensional}]$ representation are given by $\{T_a, X_b, Y_c\}$, where the iT_a , iX_b and iY_c are real antisymmetric $(2N+1) \times (2N+1)$ matrices that satisfy:

$$T_a \Sigma_0 + \Sigma_0 T_a^T = 0, \quad (37)$$

$$X_b \Sigma_0 - \Sigma_0 X_b^T = 0, \quad (38)$$

$$\Sigma_0^T Y_c \Sigma_0 = 0, \quad (39)$$

then the T_a span an unbroken $U(N)$ Lie subalgebra, while the $\{X_b, Y_c\}$ are the broken generators that span an $SO(2N+1)/U(N)$ homogeneous space. Furthermore, $\text{Tr } (T_a X_b) = \text{Tr } (T_a Y_c) = \text{Tr } (X_b Y_c) = 0$.

Proof: Here, I shall only sketch the modifications to the proof given in Section 4. Again, we easily derive $K \tilde{T}_a^T = -\tilde{T}_a K$ [where K is defined in

eq. (24)]. In the present case, we use the fact that $\tilde{T}_a$ is antisymmetric to conclude that $K\tilde{T}_a = \tilde{T}_a K$. That is, we may write the unbroken generators, $\tilde{T}_a$, in the form of eq. (27) where $t_a = -Jt_a J$ and the it_a are $2N \times 2N$ real antisymmetric matrices. Using the results previously obtained, it follows that the $\tilde{T}_a$ span a $U(N)$ subalgebra. Likewise, the broken generators satisfy: $K\tilde{X}_a = -\tilde{X}_a K$ and $K^T \tilde{Y}_c K = 0$. The antisymmetry of $\tilde{X}_a$ and $\tilde{Y}_a$ implies

$$i\tilde{X}_b = \begin{pmatrix} x_b & 0 \\ 0 & 0 \end{pmatrix}, \quad i\tilde{Y}_c = \begin{pmatrix} 0 & y_c \\ -y_c^T & 0 \end{pmatrix}, \quad (40)$$

where x_b is an $2N \times 2N$ real antisymmetric matrix that satisfies $x_b = Jx_b J$, and y_c is a real $2N$ -dimensional column vector. From the explicit forms above, it is easy to check that $\text{Tr}(\tilde{T}_a \tilde{X}_b) = \text{Tr}(\tilde{T}_a \tilde{Y}_c) = \text{Tr}(\tilde{X}_b \tilde{Y}_c) = 0$, which implies that $\text{Tr}(T_a X_b) = \text{Tr}(T_a Y_c) = \text{Tr}(X_b Y_c) = 0$.

Finally, we count the number of $SO(2N+1)$ generators $\{\tilde{T}_a, \tilde{X}_b, \tilde{Y}_c\}$. There are N^2 unbroken generators and $N(N+1)$ degrees of freedom associated with $\tilde{X}_b$ as in the case of $SO(2N)$ breaking. Finally, adding in the $2N$ parameters that describes $\tilde{Y}_c$ yields a total number of $SO(2N+1)$ generators equal to $N(2N+1)$ as expected.

References

- [1] L.-F. Li, *Phys. Rev.* **D9**, 1723 (1974); V. Elias, S. Eliezer and A.R. Swift, *Phys. Rev.* **D12**, 3356 (1975); D. Wu, *Nucl. Phys.* **B199**, 523 (1982) [E: **B213**, 545 (1983)].
- [2] R.A. Horn and C.R. Johnson, *Matrix Analysis* (Cambridge University Press, Cambridge, England, 1990).
- [3] R.N. Mohapatra and P.B. Pal, *Massive Neutrinos in Physics and Astrophysics*, 2nd edition (World Scientific, Singapore, 1998).
- [4] See problem 26 on p. 217 of ref. [2].
- [5] J.F. Cornwell, *Group Theory in Physics*, Volume 2 (Academic Press, London, 1984)